

Degree powers in graphs with a forbidden forest

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Abstract

Given a positive integer p and a graph G with degree sequence d_1, \dots, d_n , we define $e_p(G) = \sum_{i=1}^n d_i^p$. Caro and Yuster introduced a Turán-type problem for $e_p(G)$: Given a positive integer p and a graph H , determine the function $\text{ex}_p(n, H)$, which is the maximum value of $e_p(G)$ taken over all graphs G on n vertices that do not contain H as a subgraph. Clearly, $\text{ex}_1(n, H) = 2\text{ex}(n, H)$, where $\text{ex}(n, H)$ denotes the classical Turán number. Caro and Yuster determined the function $\text{ex}_p(n, P_\ell)$ for sufficiently large n , where $p \geq 2$ and P_ℓ denotes the path on ℓ vertices. In this paper, we generalise this result and determine $\text{ex}_p(n, F)$ for sufficiently large n , where $p \geq 2$ and F is a linear forest. We also determine $\text{ex}_p(n, S)$, where S is a star forest; and $\text{ex}_p(n, B)$, where B is a broom graph with diameter at most six.

Keywords: degree power; Turán-type problem; H -free; forest

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1 Introduction

For standard graph-theoretic notation and terminology, the reader is referred to [1]. All graphs considered here are finite, undirected, and have no loops or multiple edges. Let G and H be two graphs. The degree of a vertex $v \in V(G)$ and the maximum degree of G are

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denoted by $d_G(v)$ and $\Delta(G)$. We use $G \cup H$ to denote the disjoint union of G and H , and $G + H$ for the *join* of G and H , i.e., the graph obtained from $G \cup H$ by adding all edges between G and H . Let kG denote k vertex-disjoint copies of G . For $U \subset V(G)$, let $G[U]$ denote the subgraph of G induced by U . Let K_t, E_t and P_t denote the complete graph, the empty graph, and the path on t vertices, respectively. Let S_r denote the star with maximum degree r . Let M_t be the graph on t vertices with a maximum matching (i.e., $\lfloor \frac{t}{2} \rfloor$ independent edges).

Given a graph H , we say that a graph G is H -free if G does not contain H as a subgraph. The classical *Turán number*, denoted by $\text{ex}(n, H)$, is the maximum number of edges in a H -free graph on n vertices. Turán's classical result [15] states that $\text{ex}(n, K_{r+1}) = e(T_r(n))$ for $n \geq r \geq 2$, where $T_r(n)$ denotes the r -partite *Turán graph* on n vertices. Given a graph G whose degree sequence is d_1, \dots, d_n , and a positive integer p , let $e_p(G) = \sum_{i=1}^n d_i^p$. Caro and Yuster [5] introduced a Turán-type problem for $e_p(G)$: Determine the function $\text{ex}_p(n, H)$, which is the maximum value of $e_p(G)$ taken over all H -free graphs G on n vertices. Moreover, characterise the *extremal graphs*, i.e., the H -free graphs G on n vertices with $e_p(G) = \text{ex}_p(n, H)$. Clearly, we have $\text{ex}_1(n, H) = 2\text{ex}(n, H)$.

This Turán-type problem has attracted significant interest from many researchers. Caro and Yuster [5] proved that $\text{ex}_p(n, K_{r+1}) = e_p(T_r(n))$ for $p = 1, 2, 3$. The same result does not hold if r is fixed, and p and n are sufficiently large. For example, if G is the complete bipartite graph with class sizes $\lfloor \frac{n}{2} \rfloor - 1$ and $\lceil \frac{n}{2} \rceil + 1$, then we have $e_4(G) > e_4(T_2(n))$. Hence, we see that the parameter p does play a role in the value of $\text{ex}_p(n, K_{r+1})$ and the extremal graphs. Bollobás and Nikiforov further studied the function $\text{ex}_p(n, K_{r+1})$, where they allowed $p > 0$ to be real. In [2], they proved that for n sufficiently large, $\text{ex}_p(n, K_{r+1}) = e_p(T_r(n))$ for $0 < p < r$, and $\text{ex}_p(n, K_{r+1}) > (1 + \varepsilon)e_p(T_r(n))$ for $p \geq r + \lceil \sqrt{2r} \rceil$ and some $\varepsilon = \varepsilon(r) > 0$. In [3], they proved a result which gives an extension of the Erdős-Stone Theorem [8] by using $e_p(G)$ instead of the number of edges.

When considering cycles as the forbidden subgraphs, Caro and Yuster [5] proved that $\text{ex}_2(n, \mathcal{C}) = e_2(F_n)$ for sufficiently large n , where \mathcal{C} denotes the family of cycles with even length (notice the natural extension of the definition of ex_p to families of graphs), and F_n is the *friendship graph* on n vertices, i.e., F_n is obtained by taking a star on n vertices and adding a maximum matching on the set of leaves. They also showed that F_n is the unique extremal graph, and remarked that the same result also holds for $p > 2$. Nikiforov [13] proved that $\text{ex}_p(n, C_{2k+2}) = (1 + o(1))kn^p$, where C_t denotes the cycle of order t , and this settled a conjecture of Caro and Yuster. Gu et al. [11] proved that for $p \geq 1$, there exists a constant $c = c(p)$ such that the following holds: If $\text{ex}_p(n, C_5) = e_p(G)$ for some C_5 -free graph G of order n , then G is a complete bipartite graph with class sizes $cn + o(n)$ and $(1 - c)n + o(n)$.

A *linear forest* (resp. *star forest*) is a forest whose connected components are paths (resp. stars). There are many known results about the function $\text{ex}_p(n, F)$ where F is a linear forest. For the case of the classical Turán number $\text{ex}(n, F)$, one of the earliest results is the case when $F = P_\ell$ is a path. Erdős and Gallai [7] proved in 1959 that $\text{ex}(n, P_\ell) \leq (\frac{\ell}{2} - 1)n$ for $\ell \geq 2$, and if $\ell - 1$ divides n , then equality holds only for the graph with vertex-disjoint copies of $K_{\ell-1}$. Motivated by this result, Erdős and Sós [6] in 1963 made the conjecture that the same result holds for any tree, i.e., if T is a tree on $t \geq 2$ vertices, then we have $\text{ex}(n, T) \leq (\frac{t}{2} - 1)n$. This long-standing conjecture remains open, and many partial results are

known. The result of Erdős and Gallai was also sharpened by Faudree and Schelp [9], when they determined the function $\text{ex}(n, P_\ell)$ exactly as well as the extremal graphs. When F has more components, Erdős and Gallai [7] also proved that $\text{ex}(n, kP_2) = \binom{k-1}{2} + (k-1)(n-k+1)$ for $k \geq 2$ and sufficiently large n , where the unique extremal graph is $K_{k-1} + E_{n-k+1}$. Very recently, this result was extended by Bushaw and Kettle [4], who determined the function $\text{ex}(n, kP_\ell)$ for $k \geq 2$, $\ell \geq 3$ and sufficiently large n . Their result was further generalised by Lidický et al. [12], who determined the function $\text{ex}(n, F)$ for an arbitrary linear forest F and sufficiently large n . In these two results, the extremal graph is unique. Lidický et al. [12] also determined the function $\text{ex}(n, S)$ for an arbitrary star forest S and sufficiently large n , and characterised the extremal graphs.

On the other hand, Caro and Yuster [5] determined the function $\text{ex}_p(n, P_\ell)$ for $p \geq 2$, $\ell \geq 3$ and sufficiently large n . The extremal graph is again unique, and is significantly different to the extremal graphs of $\text{ex}(n, P_\ell)$ obtained by Faudree and Schelp [9]. They also determined the functions $\text{ex}_p(n, S_r)$ and $\text{ex}_p(n, S_r^*)$, and their extremal graphs, where S_r^* is the graph obtained by attaching a pendent edge at a leaf of S_r .

This paper will be organised as follows. In Section 2, we will state precisely the previously known results about the function $\text{ex}_p(n, F)$, for various forests F . In Sections 3 and 4, we will determine the function $\text{ex}_p(n, F)$ when F is a linear forest, a star forest, and a broom with diameter at most 6 (A *broom* is a path with a star attached at one end). Our results can be regarded as extensions to many of these previously known results from [4, 5, 7, 12]. Unless otherwise stated, we assume that n is always sufficiently large, and we will make no serious attempt to minimise the lower bound on n . Without going into details, we remark that every large lower bound on n depends only on the forest F , and not the parameter p .

2 Known results

In this section, we will review many of the known results about the function $\text{ex}_p(n, F)$, for various forests F . Some of these results will also be helpful for us to present our results in Sections 3 and 4. First, we collect the results where F is a single component. When F is a path, Caro and Yuster [5] observed that for $p \geq 1$, we have

$$\text{ex}_p(n, P_2) = 0, \quad \text{and} \quad \text{ex}_p(n, P_3) = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases} \quad (1)$$

Moreover, the unique extremal graph for $\text{ex}_p(n, P_3)$ is M_n , the graph on n vertices with a maximum matching. For $F = P_\ell$, Erdős and Gallai [7] proved the following result, as we have mentioned in the introduction.

Theorem 1. [7] *For $\ell \geq 2$, we have $\text{ex}(n, P_\ell) \leq (\frac{\ell}{2} - 1)n$. Moreover, if $\ell - 1$ divides n , then equality holds only for the graph with vertex-disjoint copies of $K_{\ell-1}$.*

Inspired by Theorem 1, Erdős and Sós [6] made the conjecture that the same result holds for any tree: If T is a tree on $t \geq 2$ vertices, then $\text{ex}(n, T) \leq (\frac{t}{2} - 1)n$. This long-standing conjecture remains open, and many partial results are known. Theorem 1 was subsequently sharpened by Faudree and Schelp [9], when they managed to determine $\text{ex}(n, P_\ell)$ exactly, as well as all the extremal graphs.

Theorem 2. [9] Let $\ell \geq 2$ and $n = a(\ell - 1) + b$, where $a \geq 0$ and $0 \leq b < \ell - 1$. We have

$$\text{ex}(n, P_\ell) = a \binom{\ell - 1}{2} + \binom{b}{2}.$$

Moreover, the extremal graphs are:

- $aK_{\ell-1} \cup K_b$,
- $a'K_{\ell-1} \cup (K_{\ell/2-1} + E_{\ell/2+(a-a'-1)(\ell-1)+b})$, where ℓ is even, $a > 0$, $b = \frac{\ell}{2}$ or $b = \frac{\ell}{2} - 1$ and $0 \leq a' < a$.

Caro and Yuster [5] determined the function $\text{ex}_p(n, P_\ell)$ for $p \geq 2$, $\ell \geq 4$, and sufficiently large n , and they showed that the extremal graph is unique. To state their result, we define the graph $H(n, \ell)$ as follows. Let $b = \lfloor \frac{\ell}{2} \rfloor - 1$. Then $H(n, \ell) = K_b + E_{n-b}$ if ℓ is even, and $H(n, \ell)$ is $K_b + E_{n-b}$ with an edge added to E_{n-b} if ℓ is odd.

Theorem 3. [5] Let $p \geq 2$, $\ell \geq 4$, and $n \geq n_0(\ell)$ be sufficiently large. Then

$$\begin{aligned} \text{ex}_p(n, P_\ell) &= e_p(H(n, \ell)) \\ &= \begin{cases} b(n-1)^p + (n-b-2)b^p + 2(b+1)^p & \text{if } \ell \text{ is odd,} \\ b(n-1)^p + (n-b)b^p & \text{if } \ell \text{ is even,} \end{cases} \end{aligned}$$

where $b = \lfloor \frac{\ell}{2} \rfloor - 1$. Moreover, $H(n, \ell)$ is the unique extremal graph.

They remarked that the extremal graph $H(n, \ell)$ for $\text{ex}_p(n, P_\ell)$, with $p \geq 2$, is very different from the extremal graphs for $\text{ex}(n, P_\ell)$ in Theorem 2. This is because $H(n, \ell)$ has large maximum degree, which plays a role in making the value of $e_p(H(n, \ell))$ large, when $p \geq 2$.

When $F = S_r$ is a star, Caro and Yuster [5] made the observation that $\text{ex}_p(n, S_r)$ is attained by a graph L on n vertices which is an extremal graph for $\text{ex}(n, S_r)$. Clearly if $n \leq r - 1$, we have $L = K_n$. For $n \geq r$, we have L is an $(r - 1)$ -regular graph if $(r - 1)n$ is even, and L has $n - 1$ vertices of degree $r - 1$ and one vertex of degree $r - 2$ if $(r - 1)n$ is odd. We call such a graph L a *near $(r - 1)$ -regular graph*, since L is as close to being $(r - 1)$ -regular as possible. It is well-known and easy to show that such graphs L exist. Note that we have $e(L) = \lfloor \frac{(r-1)n}{2} \rfloor$. Thus, the observation of Caro and Yuster is the following.

Proposition 4. [5] Let $p \geq 1$, and let S_r be the star with maximum degree $r \geq 1$.

- (a) If $n \leq r - 1$, then $\text{ex}_p(n, S_r) = n(n - 1)^p$. Moreover, the unique extremal graph is K_n .
- (b) If $n \geq r$, then

$$\text{ex}_p(n, S_r) = \begin{cases} (n-1)(r-1)^p + (r-2)^p & \text{if } (r-1)n \text{ is odd,} \\ n(r-1)^p & \text{if } (r-1)n \text{ is even.} \end{cases}$$

Moreover, the extremal graphs are the near $(r - 1)$ -regular graphs on n vertices.

For $\ell \geq 4$ and $s \geq 0$, let $B_{\ell,s}$ be the graph on $\ell + s$ vertices, obtained by adding s pendent edges to a penultimate vertex v of P_ℓ . Such a graph $B_{\ell,s}$ is a *broom*, and v is the *centre* of the broom.

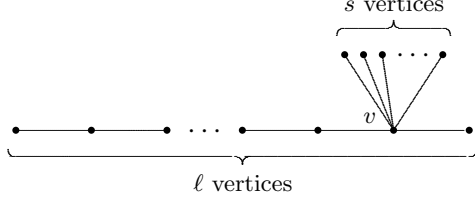


Figure 1. The broom graph $B_{\ell,s}$

It is interesting to study Turán-type problems for brooms, because a broom may be considered as a generalisation of both a path and a star. Sun and Wang [14] determined the function $\text{ex}(n, B_{4,s})$ for $s \geq 1$, as follows.

Theorem 5. [14] *Let $s \geq 1$ and $n \geq s+4$. Let $n = a(s+3)+b$, where $a \geq 1$ and $0 \leq b < s+3$. We have*

$$\text{ex}(n, B_{4,s}) = \begin{cases} (a-1) \binom{s+3}{2} + \left\lfloor \frac{(s+1)(s+3+b)}{2} \right\rfloor & \text{if } s \geq 3 \text{ and } 2 \leq b \leq s, \\ a \binom{s+3}{2} + \binom{b}{2} & \text{otherwise.} \end{cases}$$

Roughly speaking, in Theorem 5, the value of $\text{ex}(n, B_{4,s})$ is attained as follows. If b is close to either 0 or $s+3$, then we would take the graph $aK_{s+3} \cup K_b$. Otherwise, we would take a graph $(a-1)K_{s+3} \cup L$, where L is a near $(s+1)$ -regular graph on $s+3+b$ vertices.

Sun and Wang also determined the function $\text{ex}(n, B_{5,s})$ for $s \geq 1$ and $n \geq s+5$. However, their result is complicated to state in full. A key result that they proved is the following.

Theorem 6. [14] *Let $s \geq 1$ and $n \geq s+5$. Let $n = a(s+4)+b$, where $a \geq 1$ and $0 \leq b < s+4$. We have*

$$\text{ex}(n, B_{5,s}) = \begin{cases} (a-1) \binom{s+4}{2} + \text{ex}(s+4+b, B_{5,s}) & \text{if } 1 \leq b \leq s, \\ a \binom{s+4}{2} + \binom{b}{2} & \text{if } b \in \{0, s+1, s+2, s+3\}. \end{cases}$$

Similarly, in Theorem 6, the value of $\text{ex}(n, B_{5,s})$ is attained by $aK_{s+4} \cup K_b$ if b is either 0 or close to $s+4$. Otherwise, we would take a graph $(a-1)K_{s+4} \cup L$, where L is an extremal graph for $B_{5,s}$ on $s+4+b$ vertices.

Caro and Yuster [5] determined the function $\text{ex}_p(n, B_{4,s})$, for $p \geq 2$ and sufficiently large n . They remarked that the result is very different to Proposition 4, even though $B_{4,s}$ is very close to being a star.

Proposition 7. [5] *Let $p \geq 2$, $s \geq 1$, and $n > 2(s+4)$. Then $\text{ex}_p(n, B_{4,s}) = e_p(S_{n-1}) = (n-1)^p + (n-1)$. Moreover, S_{n-1} is the unique extremal graph.*

Now we consider the case when the forest F has more than one component. When $F = kP_2$, the classical Turán number $\text{ex}(n, kP_2)$ was determined by Erdős and Gallai [7].

Theorem 8. [7] *Let $k \geq 2$ and $n > \frac{5k}{2} - 1$. We have $\text{ex}(n, kP_2) = \binom{k-1}{2} + (k-1)(n-k+1)$. Moreover, $K_{k-1} + E_{n-k+1}$ is the unique extremal graph.*

For $n \leq \frac{5k}{2} - 1$, Erdős and Gallai also determined $\text{ex}(n, kP_2)$ and the extremal graphs, which are different from those in Theorem 8. For the function $\text{ex}(n, kP_3)$, Yuan and Zhang [16] obtained the following result.

Theorem 9. [16] *Let $k \geq 2$ and $n > 5k - 1$. We have $\text{ex}(n, kP_3) = \binom{k-1}{2} + (k-1)(n-k+1) + \lfloor \frac{n-k+1}{2} \rfloor$. Moreover, $K_{k-1} + M_{n-k+1}$ is the unique extremal graph.*

In fact, Yuan and Zhang completely determined $\text{ex}(n, kP_3)$ and the extremal graphs for all n , which solved a conjecture of Gorgol [10]. Bushaw and Kettle [4] had previously proved the case of Theorem 9 for $n \geq 7k$.

Next, there are results for the case when $F = \bigcup_{i=1}^k P_{\ell_i}$ is a linear forest, where $k \geq 2$, and we may assume that $\ell_1 \geq \dots \geq \ell_k \geq 2$. To describe the results, we define the graph $H(n, F)$ as follows. Let $b = \sum_{i=1}^k \lfloor \frac{\ell_i}{2} \rfloor - 1$. Then, $H(n, F)$ is $K_b + E_{n-b}$ with a single edge added to E_{n-b} if all ℓ_i are odd, and $H(n, F) = K_b + E_{n-b}$ otherwise. Note that $H(n, F)$ is F -free. Indeed, if $H(n, F)$ contains F , then the path in F of order ℓ_i must use at least $\lfloor \frac{\ell_i}{2} \rfloor$ vertices of the K_b . But this cannot happen for every path in F , by the definition of b .

In the case when $F = kP_\ell$, we write $H(n, k, \ell)$ for $H(n, F)$. We have already seen the results for $\text{ex}(n, kP_\ell)$ when $\ell = 2, 3$ (Theorems 8 and 9). For $\ell \geq 4$, Bushaw and Kettle [4] proved the following result.

Theorem 10. [4] *Let $k \geq 2$, $\ell \geq 4$, and $n \geq 2\ell + 2k\ell(\lceil \frac{\ell}{2} \rceil + 1)\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$. We have*

$$\text{ex}(n, kP_\ell) = e(H(n, k, \ell)) = \binom{k\lfloor \frac{\ell}{2} \rfloor - 1}{2} + \left(k\lfloor \frac{\ell}{2} \rfloor - 1\right)\left(n - k\lfloor \frac{\ell}{2} \rfloor + 1\right) + c,$$

where $c = 1$ if ℓ is odd, and $c = 0$ if ℓ is even. Moreover, $H(n, k, \ell)$ is the unique extremal graph.

This result was extended by Lidický et al. [12], who determined $\text{ex}(n, F)$ for an arbitrary linear forest $F \neq kP_3$.

Theorem 11. [12] *Let $k \geq 2$, and $F = \bigcup_{i=1}^k P_{\ell_i}$ be a linear forest, where $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k \geq 2$ and $\ell_i \neq 3$ for some i . Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\text{ex}(n, F) = e(H(n, F)) = \binom{\sum_{i=1}^k \lfloor \frac{\ell_i}{2} \rfloor - 1}{2} + \left(\sum_{i=1}^k \lfloor \frac{\ell_i}{2} \rfloor - 1\right)\left(n - \sum_{i=1}^k \lfloor \frac{\ell_i}{2} \rfloor + 1\right) + c,$$

where $c = 1$ if all ℓ_i are odd, and $c = 0$ otherwise. Moreover, $H(n, F)$ is the unique extremal graph.

Finally, Lidický et al. [12] determined the function $\text{ex}(n, F)$, when F is a star forest and n is sufficiently large. Let $F = \bigcup_{i=1}^k S_{r_i}$, where $r_1 \geq \dots \geq r_k \geq 1$. To describe their result, we define a graph $G(n, F)$ as follows. Let $i, r \geq 1$, and L be a graph on $n - i + 1$ vertices which is an extremal graph for S_r . Thus L is a near $(r - 1)$ -regular graph, and $e(L) = \lfloor \frac{r-1}{2}(n - i + 1) \rfloor$. Let $G(n, i, r) = K_{i-1} + L$. Now, let $G(n, F)$ be any graph $G(n, i, r_i)$ where $e(G(n, i, r_i))$ is maximised over $1 \leq i \leq k$. Note that each of $G(n, i, r_i)$ and $G(n, F)$ can be one of many possible graphs.

Observe that $G(n, i, r_i)$ is F -free for all $1 \leq i \leq k$. Indeed, if $G(n, i, r_i) = K_{i-1} + L$ as defined and contains a copy of F , then each star $S_{r_1}, \dots, S_{r_{i-1}}$ must have at least one vertex from the K_{i-1} , and S_{r_i} is not a subgraph of L .

Lidický et al. [12] proved that the graphs $G(n, F)$ are extremal for F .

Theorem 12. [12] *Let $k \geq 2$, and $F = \bigcup_{i=1}^k S_{r_i}$ be a star forest, where $r_1 \geq \dots \geq r_k \geq 1$ are the maximum degrees of the components. Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\begin{aligned} \text{ex}(n, F) &= e(G(n, F)) \\ &= \max_{1 \leq i \leq k} \left\{ (i-1)(n-i+1) + \binom{i-1}{2} + \left\lfloor \frac{r_i-1}{2}(n-i+1) \right\rfloor \right\}. \end{aligned}$$

Moreover, the extremal graphs are the graphs $G(n, F)$.

3 Linear and star forests

We now study the function $\text{ex}_p(n, F)$, where F is a linear forest or a star forest, $p \geq 2$, and n is sufficiently large. We assume throughout this section that F has at least two components, since the single component case is covered by (1), Theorem 3, and Proposition 4.

We first consider the case when F is a star forest. Recall that S_r is the star with maximum degree r . Let $F = \bigcup_{i=1}^k S_{r_i}$, where $r_1 \geq \dots \geq r_k \geq 1$. Our first result is the following. It turns out that $\text{ex}_p(n, F)$ is attained by the graphs $G(n, k, r_k)$.

Theorem 13. *Let $k, p \geq 2$, and $F = \bigcup_{i=1}^k S_{r_i}$ be a star forest, where $r_1 \geq \dots \geq r_k \geq 1$ are the maximum degrees of the components. Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\text{ex}_p(n, F) = e_p(G(n, k, r_k)).$$

Moreover, the extremal graphs are the graphs $G(n, k, r_k)$.

Proof. Since $G(n, k, r_k)$ does not contain a copy of F , we have $\text{ex}_p(n, F) \geq e_p(G(n, k, r_k))$. To prove the theorem, it suffices to show that any F -free graph G on n vertices with $G \neq G(n, k, r_k)$ satisfies $e_p(G) < e_p(G(n, k, r_k))$.

It is easy to calculate that

$$\begin{aligned} e_p(G(n, k, r_k)) &= \begin{cases} (k-1)(n-1)^p + (n-k+1)(r_k+k-2)^p & \text{if one of } r_k-1 \text{ and } n-k+1 \text{ is even} \\ (k-1)(n-1)^p + (n-k)(r_k+k-2)^p + (r_k+k-3)^p & \text{if } r_k-1 \text{ and } n-k+1 \text{ are odd} \end{cases} \\ &= (k-1)n^p + o(n^p). \end{aligned} \tag{2}$$

We may assume that there exists a subset $U \subset V(G)$ of order $k-1$ such that every vertex $v \in U$ has degree $d_G(v) \geq \sum_{i=1}^k r_i + k$. Otherwise, if G has at most $k-2$ such vertices, then using $p \geq 2$ and (2), we have

$$\begin{aligned} e_p(G) &< (k-2)n^p + (n-k+2) \left(\sum_{i=1}^k r_i + k \right)^p < (k-1)n^p + o(n^p) \\ &= e_p(G(n, k, r_k)). \end{aligned}$$

Now, we prove that $G \subset G(n, k, r_k)$, which implies that $e_p(G) < e_p(G(n, k, r_k))$. Recall that $G(n, k, r_k) = K_{k-1} + L$, where L is a graph on $n-k+1$ vertices which is an extremal graph for S_{r_k} . Thus by identifying U with K_{k-1} and $G-U$ with L , we have $G \subset G(n, k, r_k)$ if we can show that $G-U$ is S_{r_k} -free. Suppose that there is a copy of S_{r_k} in $G-U$. Then, using the fact that $d_G(v) \geq \sum_{i=1}^k r_i + k$ for every $v \in U$, we can find vertex-disjoint copies of $S_{r_1}, \dots, S_{r_{k-1}}$, using vertices in U as their centres, and with their neighbours in $G-U$, not using the vertices of the S_{r_k} , as leaves. This gives a copy of F in G , a contradiction. \square

Now, we consider the case when F is an arbitrary linear forest. Let $F = \bigcup_{i=1}^k P_{\ell_i}$, where $k \geq 2$ and $\ell_1 \geq \dots \geq \ell_k \geq 2$. For the case when $F = kP_3$, we can set $r_1 = \dots = r_k = 2$ in Theorem 13 to obtain the following result, which can be considered as an extension to Theorem 9.

Corollary 14. *Let $k, p \geq 2$ and $n \geq n_0(k)$ be sufficiently large. We have $\text{ex}_p(n, kP_3) = e_p(K_{k-1} + M_{n-k+1})$. Moreover, $K_{k-1} + M_{n-k+1}$ is the unique extremal graph.*

Now, let $F \neq kP_3$. We shall prove the following result, which can be considered as an extension to Theorem 11.

Theorem 15. *Let $k, p \geq 2$, and $F = \bigcup_{i=1}^k P_{\ell_i}$ be a linear forest, where $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k \geq 2$ and $\ell_i \neq 3$ for some i . Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\text{ex}_p(n, F) = e_p(H(n, F)).$$

Moreover, $H(n, F)$ is the unique extremal graph.

In particular, if $F = kP_\ell$ and $\ell \neq 3$, then

$$\text{ex}_p(n, kP_\ell) = e_p(H(n, k, \ell)).$$

Moreover, $H(n, k, \ell)$ is the unique extremal graph.

Before we prove Theorem 15, we first recall a lemma of Caro and Yuster [5].

Lemma 16. [5] *Let $b \geq 1$ and $p \geq 2$ be integers. Let G be a graph on n vertices such that $e(G) \leq (b + \frac{1}{2})n$. Let $d_1 \geq \dots \geq d_n$ be the degree sequence of G . Then, if $d_b \leq 0.65n$, we have $e_p(G) \leq cn^p + O(n^{p-1})$ for some constant c with $0 < c < b$.*

Although Lemma 16 is not stated explicitly in [5], it can be seen easily in the proof of Lemma 3.5 in [5].

Proof of Theorem 15. Since $H(n, F)$ is F -free, we have $\text{ex}_p(n, F) \geq e_p(H(n, F))$. Hence, it suffices to show that any F -free graph G on n vertices with $G \neq H(n, F)$ has $e_p(G) < e_p(H(n, F))$. Assume the contrary, and let G be an F -free graph on n vertices, that is maximal in the sense that $e_p(G) = \text{ex}_p(n, F) \geq e_p(H(n, F))$ and $G \neq H(n, F)$.

Let $b = \sum_{i=1}^k \lfloor \frac{\ell_i}{2} \rfloor - 1 \geq 1$. By the definition of $H(n, F)$, it is easy to calculate that

$$\begin{aligned} e_p(H(n, F)) &= \begin{cases} b(n-1)^p + (n-b-2)b^p + 2(b+1)^p & \text{if all } \ell_i \text{ are odd} \\ b(n-1)^p + (n-b)b^p & \text{otherwise} \end{cases} \\ &= bn^p + o(n^p). \end{aligned} \quad (3)$$

According to Theorem 11, we have

$$e(G) \leq \binom{b}{2} + b(n-b) + 1 \leq bn. \quad (4)$$

Let $d_1 \geq \dots \geq d_n$ be the degree sequence of G . Let $X \subset V(G)$ be the set of vertices with degrees d_1, \dots, d_b , and $Y = V(G) \setminus X$. By Lemma 16 and using (3) and (4), we may assume that $d_b > 0.65n$. Let $A \subset Y$ be the set of vertices that have a neighbour in X , and $B = Y \setminus A$. Note that any two vertices $u, v \in X$ have at least $2(0.65n - 1) - (n - 2) = 0.3n$ common neighbours in G , and hence at least $0.3n - (b - 2) > 0.29n$ common neighbours in A . This means that for any set $Y' \subset Y$ with $|Y'|$ depending only on ℓ_1, \dots, ℓ_k (and hence $|Y'|$ is much smaller than n), u and v have a common neighbour in $A \setminus Y'$. Likewise, any vertex of X has a neighbour in $A \setminus Y'$. We now prove a series of claims.

Claim 17. *If ℓ_i is odd for all i , then $G[Y]$ does not contain a copy of P_3 with an end-vertex in A . If ℓ_i is even for some i , then $G[Y]$ does not contain an edge with an end-vertex in A .*

Proof. Suppose first that ℓ_i is odd for all i , and that $G[Y]$ contains a path $c_1c_2a_1$ with $a_1 \in A$. Let $y_1 \in X$ be a neighbour of a_1 , and $y_2 \in X \setminus \{y_1\}$. Then, y_1 and y_2 have a common neighbour $a_2 \in A \setminus \{c_1, c_2, a_1\}$. Repeating this procedure, we can obtain a path $c_1c_2a_1y_1a_2y_2 \dots y_{b-1}a_b y_b a_{b+1}$, where $X = \{y_1, \dots, y_b\}$ and $a_2, a_3, \dots, a_{b+1} \in A$. This path has $2b+3 = \ell_1 + \sum_{i=2}^k (\ell_i - 1)$ vertices, and so it contains vertex-disjoint paths $P_{\ell_1}, P_{\ell_2-1}, \dots, P_{\ell_k-1}$ with c_1 in the P_{ℓ_1} . Note that each of the paths $P_{\ell_2-1}, \dots, P_{\ell_k-1}$ has an end-vertex in X , and so we can extend each P_{ℓ_i-1} to P_{ℓ_i} by taking a neighbour of the end-vertex in X . By choosing the $k-1$ neighbours to be distinct vertices in $A \setminus \{c_1, c_2, a_1, \dots, a_{b+1}\}$, we obtain a copy of F in G , a contradiction.

Now, let $Q = \{1 \leq i \leq k : \ell_i \text{ is even}\} \neq \emptyset$, and suppose that $G[Y]$ contains an edge ca_1 with $a_1 \in A$. As before, we can obtain a path $ca_1y_1a_2y_2 \dots y_{b-1}a_by_b a_{b+1}$, where $X = \{y_1, \dots, y_b\}$ and $a_2, a_3, \dots, a_{b+1} \in A \setminus \{c, a_1\}$. This path has $2b+2 = \sum_{i \notin Q} (\ell_i - 1) + \sum_{i \in Q} \ell_i$ vertices. We obtain vertex-disjoint paths P_{ℓ_i-1} for $i \notin Q$, and P_{ℓ_i} for $i \in Q$, such that the path using c is P_{ℓ_j} , for some $j \in Q$. Extending each P_{ℓ_i-1} to P_{ℓ_i} for $i \notin Q$, we again have a copy of F in G , a contradiction. \square

Claim 18. *$G[B]$ does not contain a copy of P_{ℓ_k} .*

Proof. Suppose that $G[B]$ contains a copy of P_{ℓ_k} . Let $Q_0 = \{1 \leq i \leq k-1 : \ell_i \text{ is even}\}$ and $Q_1 = \{1 \leq i \leq k-1 : \ell_i \text{ is odd}\}$. We can find a path $a_1y_1a_2y_2 \cdots a_by_b$, where $X = \{y_1, \dots, y_b\}$ and $a_1, \dots, a_b \in A$. This path has $2b \geq \sum_{i \in Q_0} \ell_i + \sum_{i \in Q_1} (\ell_i - 1)$ vertices, and hence contains vertex-disjoint paths P_{ℓ_i} for $i \in Q_0$, and P_{ℓ_i-1} for $i \in Q_1$. Extending each P_{ℓ_i-1} to P_{ℓ_i} for $i \in Q_1$, we have a copy of F in G , a contradiction. \square

Claim 19. $B = \emptyset$.

Proof. Suppose that $B \neq \emptyset$. If $\ell_k = 2$, then note that Claims 17 and 18 imply that $G[Y]$ does not contain any edges. This means that G is a subgraph of $H(n, F)$, and we have $e_p(G) < e_p(H(n, F))$, a contradiction. Now, let $\ell_k \geq 3$. We will derive a contradiction by constructing a new F -free graph G' such that $e_p(G') > e_p(G)$.

Note that by Claim 17, every vertex of B has at most one neighbour in G lying in A . Since Claim 18 implies that $G[B]$ is P_{ℓ_k} -free, by Theorem 1, $G[B]$ contains at most $(\frac{\ell_k}{2} - 1)|B|$ edges. Hence, there exists a vertex $v \in B$ with at most $\ell_k - 2$ neighbours in $G[B]$. Now in G , in view of Claim 17, one of the following holds.

- (i) $d_G(v) = 1$, with the only neighbour of v , say u , lying in A .
- (ii) $0 \leq d_G(v) \leq \ell_k - 2$, with all neighbours of v lying in B .

Delete all edges adjacent to v in G , connect v to all vertices of X , and denote the new graph by G' . We claim that G' is also F -free. Indeed, if G' contains a copy of F , then exactly one path of F , say P_{ℓ_j} , must use an edge vy_1 , for some $y_1 \in X$. If v is not an end-vertex of such a P_{ℓ_j} , then the P_{ℓ_j} also contains another neighbour $y_2 \in X$ of v . We can find a common neighbour $v' \in A$ of y_1 and y_2 in G which is not used in the copy of F . If v is an end-vertex of the P_{ℓ_j} , then we take $v' \in A$ to be any neighbour of y_1 not in the copy of F . Replacing v with v' on the P_{ℓ_j} , we obtain a copy of F in G , a contradiction.

We now show that $e_p(G') > e_p(G)$. Consider the effect of the transformation from G to G' on the degree sequence. The degrees of the vertices of X have increased by one. The degree of v has not decreased, since $d_{G'}(v) - d_G(v) \geq b - (\ell_k - 2) \geq (2\lfloor \frac{\ell_k}{2} \rfloor - 1) - (\ell_k - 2) \geq 0$. The degrees of the neighbours of v in G have decreased by 1. Since every vertex of X has degree at least $0.65n$, the total increase in $e_p(G') - e_p(G)$ contributed by the vertices of X is at least

$$b((0.65n + 1)^p - (0.65n)^p) = bp(0.65n)^{p-1} + o(n^{p-1}).$$

Also, $d_b > 0.65n$ implies that $|B| < 0.35n$. If (i) holds, then Claim 17 implies that in G , u has no neighbours in A , and hence $d_G(u) < 0.35n + b < 0.36n$. The decrease in $e_p(G') - e_p(G)$ contributed by u is at most

$$(0.36n)^p - (0.36n - 1)^p = p(0.36n)^{p-1} + o(n^{p-1}).$$

Suppose that (ii) holds. Then in G , every neighbour of v (lying in B) cannot have a neighbour in A , in view of Claim 17. Hence, every neighbour of v has degree at most $0.35n$ in G . The total decrease in $e_p(G') - e_p(G)$ contributed by the neighbours of v is at most

$$(\ell_k - 2)((0.35n)^p - (0.35n - 1)^p) \leq (\ell_k - 2)p(0.36n)^{p-1} + o(n^{p-1}).$$

Since $b \geq 2\lfloor \frac{\ell_k}{2} \rfloor - 1 \geq \ell_k - 2$, we have

$$\begin{aligned} e_p(G') - e_p(G) &\geq p \left(b \left(\frac{0.65}{0.36} \right)^{p-1} - (\ell_k - 2) \right) 0.36^{p-1} n^{p-1} + o(n^{p-1}) \\ &\geq 0.72(1.8(\ell_k - 2) - (\ell_k - 2)) n^{p-1} + o(n^{p-1}) \\ &\geq 0.576 n^{p-1} + o(n^{p-1}) > 0. \end{aligned}$$

□

By Claim 19, we may assume that $A = Y$ for the rest of the proof.

Claim 20. *If ℓ_i is odd for all i , then $G[Y]$ contains at most one edge. If ℓ_i is even for some i , then $G[Y]$ does not contain an edge.*

Proof. The latter assertion follows immediately from Claim 17 and the fact that $A = Y$. Now, assume that ℓ_i is odd for all i . Then, since we do not have $\ell_1 = \dots = \ell_k = 3$, we have $\ell_1 \geq 5$. Assume the contrary. Together with Claim 17, the subgraph $G[Y]$ has at least two edges, and its components are single edges and isolated vertices. We consider three cases.

Case 1.

- *Either $\ell_1 = 5$, and there are two edges c_1c_2, c_3c_4 in $G[Y]$, with c_2, c_3 having a common neighbour in X ;*
- *Or $\ell_1 \geq 7$, and there are two edges c_1c_2, c_3c_4 in $G[Y]$, with c_2, c_3 having distinct neighbours in X .*

Let $q = \lfloor \frac{\ell_1}{2} \rfloor - 1$. For $\ell_1 = 5$, let c_2, c_3 have a common neighbour $y_1 \in X$, so that $c_1c_2y_1c_3c_4$ is a copy of P_5 . For $\ell_1 \geq 7$, let c_2, c_3 have distinct neighbours $y_1, y_q \in X$. Then, as before, we can find a copy of P_{ℓ_1} in the form $c_1c_2y_1a_2y_2 \dots a_qy_qc_3c_4$, where $y_2, \dots, y_{q-1} \in X$ and $a_2, \dots, a_q \in Y$. In both cases, we can again find a path $a_{q+1}y_{q+1}a_{q+2}y_{q+2} \dots a_by_b$, where $X = \{y_1, \dots, y_b\}$ and $a_{q+1}, a_{q+2}, \dots, a_b \in Y \setminus \{c_1, c_2, c_3, c_4, a_2, \dots, a_q\}$. This path has $2(b - q) = \sum_{i=2}^k (\ell_i - 1)$ vertices, and so contains vertex-disjoint copies of $P_{\ell_2-1}, \dots, P_{\ell_k-1}$. As before, we can extend these to copies of $P_{\ell_2}, \dots, P_{\ell_k}$ so that we have a copy of F in G , a contradiction.

Case 2. $\ell_1 = 5$, and no two vertices from distinct edges in $G[Y]$ have a common neighbour in X .

We shall prove that $e_p(G) < e_p(H(n, F))$, which will contradict the choice of G . Recall that $H(n, F)$ is $K_b + E_{n-b}$ with an edge uv added to the empty class, and note that $b \geq \lfloor \frac{5}{2} \rfloor + \lfloor \frac{3}{2} \rfloor - 1 = 2$. Let $d_H(z)$ denote the degree of a vertex z in $H(n, F)$. Clearly, for every $z \in X$ and every vertex z' in the K_b , we have $d_G(z) \leq n - 1 = d_H(z')$. Now, let $u_1u_2, \dots, u_{2s-1}u_{2s}$ be all the independent edges in $G[Y]$, for some $s \geq 2$, and let Γ_i be the set of vertices in X that are adjacent to at least one of u_{2i-1} and u_{2i} , for $i = 1, \dots, s$. We may assume that $|\Gamma_1| \geq \dots \geq |\Gamma_s| \geq 1$. Note that the Γ_i are pairwise disjoint subsets of X , so that $|\Gamma_2| \leq \frac{b}{2}$. Also, since u_3 has a neighbour in X , we have $|\Gamma_1| \leq b - 1$. Hence, $d_G(u_j) \leq b < d_H(u) = d_H(v)$ for $j = 1, 2$; and for every $z' \neq u, v$ in the E_{n-b} , we have $d_G(u_j) \leq \frac{b}{2} + 1 \leq b = d_H(z')$ for $j = 3, \dots, 2s$; and $d_G(z) \leq b = d_H(z')$ for $z \in Y \setminus \{u_1, \dots, u_{2s}\}$. The degree sequence of

$H(n, F)$ strictly majorises that of G , and therefore, we have $e_p(G) < e_p(H(n, F))$.

Case 3. $\ell_1 \geq 7$, and all vertices of the edges of $G[Y]$ are connected to a single vertex of X .

Let $y_1 \in X$ be this single vertex, and note that $\ell_1 \geq 7$ implies that $b \geq 3$. We construct an F -free graph G' on n vertices such that $e_p(G') > e_p(G)$, which contradicts the choice of G . Let $y_2 \in X \setminus \{y_1\}$, and let Y^* denote the set of non-isolated vertices in $G[Y]$. Observe that $|Y^*| \geq 4$ and no vertex of Y^* is adjacent to any vertex of $X \setminus \{y_1\}$. We construct G' as follows: delete the $\lfloor \frac{|Y^*|}{2} \rfloor$ independent edges of $G[Y]$, and join y_2 to each vertex of Y^* .

Similar to Claim 19, we see that G' is an F -free graph. Indeed, if G' contains a copy of F , then exactly one path, say P_{ℓ_j} , must use an edge vy_2 , for some $v \in Y^*$. If v is not an end-vertex of such a P_{ℓ_j} , then the other neighbour of v in the P_{ℓ_j} is y_1 . Now in G , we can find a common neighbour $v' \in Y$ of y_1 and y_2 which is not used in the copy of F . If v is an end-vertex of the P_{ℓ_j} , then we can take $v' \in Y$ to be any neighbour of y_2 not in the copy of F . Replacing v with v' on the P_{ℓ_j} , we obtain a copy of F in G , a contradiction.

However, the degree sequence of G' strictly majorises that of G , since the degree of y_2 has strictly increased, and all other degrees have not changed. Hence $e_p(G') > e_p(G)$, which is the required contradiction. \square

By Claim 20, G is a spanning subgraph of $H(n, F)$. Hence $e_p(G) < e_p(H(n, F))$, which contradicts the choice of G . The proof of Theorem 15 is complete. \square

4 Brooms

In this section, we shall consider the function $\text{ex}_p(n, B_{\ell, s})$, where $B_{\ell, s}$ is a broom graph, $p \geq 2$, $\ell \geq 4$, $s \geq 0$, and n is sufficiently large. As we have already seen, Theorems 5 and 6 appear to suggest that the determination of the Turán function $\text{ex}(n, B_{\ell, s})$ and the corresponding extremal graphs may be a complicated problem, in the sense that the potential results may be difficult to state. Somewhat surprisingly, we shall see here that the same problem for $\text{ex}_p(n, B_{\ell, s})$, where $p \geq 2$, may possibly be more manageable. Since the case $\ell = 4$ is covered in Proposition 7, we consider $\ell \geq 5$. Here, we will provide the answers for the cases $\ell = 5, 6, 7$, and present a conjecture for the case of general ℓ . The case $\ell = 5$ turns out to be a rather special case. Although the case $s = 0$ is covered by Theorem 3, we will include this case here since we will obtain some explicit lower bounds for n .

Theorem 21. *Let $p \geq 2$, $s \geq 0$, and $n > (2s + 10)^2$. We have*

$$\text{ex}_p(n, B_{5, s}) = \begin{cases} e_p(H(n, 5)) & \text{if } s = 0, \\ e_p(K_1 + M_{n-1}) & \text{if } s \geq 1. \end{cases}$$

Moreover, the unique extremal graph is $H(n, 5)$ if $s = 0$, and $K_1 + M_{n-1}$ if $s \geq 1$.

Theorem 22. *Let $p \geq 2$, $s \geq 0$, and $n > (2s + 12)^2$. We have*

$$\text{ex}_p(n, B_{6, s}) = e_p(H(n, 6)).$$

Moreover, $H(n, 6)$ is the unique extremal graph.

Theorem 23. Let $p \geq 2$, $s \geq 0$, and $n > (3s + 31)^2$. We have

$$\text{exp}_p(n, B_{7,s}) = e_p(H(n, 7)).$$

Moreover, $H(n, 7)$ is the unique extremal graph.

In view of Theorems 22 and 23, we believe that the following assertion may be true.

Conjecture 24. Let $p \geq 2$, $\ell \geq 6$, $s \geq 0$, and $n \geq n_0(\ell, s)$ be sufficiently large. We have

$$\text{exp}_p(n, B_{\ell,s}) = e_p(H(n, \ell)).$$

Moreover, $H(n, \ell)$ is the unique extremal graph.

That is, Conjecture 24 claims that if n is sufficiently large, then $\text{exp}_p(n, B_{\ell,s})$ is exactly the same as $\text{exp}_p(n, P_\ell)$, with the same unique extremal graph $H(n, \ell)$. If Conjecture 24 is true, then it can be considered as an extension to Theorem 3.

Furthermore, we may also ask for the situation for a *broom forest*, which is a forest whose components are brooms and stars.

Problem 25. Let $p \geq 2$, and F be a broom forest. For sufficiently large $n \geq n_0(F)$, determine the function $\text{exp}_p(n, F)$ and the extremal graphs.

Before we prove Theorems 21 to 23, we first prove some auxiliary lemmas.

Lemma 26. Let $p \geq 2$, $n_1, n_2 \geq \ell$, and $n = n_1 + n_2$.

- (a) If $\ell = 5$, then $e_p(K_1 + M_{n_1-1}) + e_p(K_1 + M_{n_2-1}) < e_p(K_1 + M_{n-1})$.
- (b) If $\ell \geq 5$, then $e_p(H(n_1, \ell)) + e_p(H(n_2, \ell)) < e_p(H(n, \ell))$.

Proof. (a) Let $\ell = 5$. Then

$$\begin{aligned} e_p(K_1 + M_{n_1-1}) + e_p(K_1 + M_{n_2-1}) &\leq (n_1 - 1)^p + (n_1 - 1)2^p \\ &\quad + (n_2 - 1)^p + (n_2 - 1)2^p \\ &< (n - 1)^p + (n - 2)2^p < e_p(K_1 + M_{n-1}). \end{aligned}$$

(b) Let $\ell \geq 5$, and $b = \lfloor \frac{\ell}{2} \rfloor - 1 \geq 1$. Since

$$\begin{aligned} e_p(H(n_1, \ell)) + e_p(H(n_2, \ell)) &\leq b(n_1 - 1)^p + (n_1 - b - 2)b^p + 2(b + 1)^p \\ &\quad + b(n_2 - 1)^p + (n_2 - b - 2)b^p + 2(b + 1)^p, \\ e_p(H(n, \ell)) &\geq b(n - 1)^p + (n - b)b^p, \end{aligned}$$

it suffices to prove that

$$b(n - 1)^p > b[(n_1 - 1)^p + (n_2 - 1)^p] + 4(b + 1)^p - (b + 4)b^p. \quad (5)$$

Clearly, $n \geq 2\ell \geq 4b + 4$. We have

$$\begin{aligned} b(n - 1)^p &> b(n - 2)^p + bp(n - 2)^{p-1} \\ &> b[(n_1 - 1)^p + (n_2 - 1)^p] + 2b(4b + 2)^{p-1}, \end{aligned}$$

which implies (5), since it is easy to verify that $2b(4b + 2)^{p-1} > 4(b + 1)^p - (b + 4)b^p$. \square

Lemma 27. Let $p \geq 2$, $\ell \geq 5$, $s \geq 0$. Let G^* be a graph on $h^* > 0$ vertices with $\Delta(G^*) \leq d = d(\ell, s)$. Let $n = h + h^* > (\ell + s + d)^2$ for some $h \geq \ell$.

(a) If $\ell = 5$, then $e_p(K_1 + M_{h-1}) + e_p(G^*) < e_p(K_1 + M_{n-1})$.

(b) If $\ell \geq 5$, then $e_p(H(h, \ell)) + e_p(G^*) < e_p(H(n, \ell))$.

Proof. (a) Let $\ell = 5$. We have

$$\begin{aligned} e_p(K_1 + M_{h-1}) + e_p(G^*) &\leq (h-1)^p + (h-1)2^p + h^*d^p, \\ e_p(K_1 + M_{n-1}) &> (n-1)^p + (n-2)2^p. \end{aligned}$$

Since $(n-2)2^p \geq (h-1)2^p$, and

$$(n-1)^p - (h-1)^p > (n-h)(n-1)^{p-1} > h^*d^{2p-2} \geq h^*d^p,$$

it follows that $e_p(K_1 + M_{h-1}) + e_p(G^*) < e_p(K_1 + M_{n-1})$.

(b) Let $\ell \geq 5$, and $b = \lfloor \frac{\ell}{2} \rfloor - 1 \geq 1$. Since

$$\begin{aligned} e_p(H(h, \ell)) + e_p(G^*) &\leq b(h-1)^p + (h-b-2)b^p + 2(b+1)^p + h^*d^p, \\ e_p(H(n, \ell)) &\geq b(n-1)^p + (n-b)b^p, \end{aligned}$$

and $(n-b)b^p > (h-b-2)b^p$, it suffices to prove that

$$b[(n-1)^p - (h-1)^p] > 2(b+1)^p + h^*d^p.$$

Clearly, $\ell + s \geq \ell \geq 2b + 2$. We have

$$\begin{aligned} b[(n-1)^p - (h-1)^p] &> (n-h)(n-1)^{p-1} \geq h^*(\ell + s + d)^{2p-2} \\ &\geq h^*(2b+2+d)^p > h^*(2b+2)^p + h^*d^p \\ &> 2(b+1)^p + h^*d^p, \end{aligned}$$

as required. \square

Before we prove the next lemma, we make some definitions. Let C be a connected graph, and $v, x \in V(C)$.

- For $y \in V(C - \{v, x\})$, the edge $e = xy \in E(C)$ is an *x-pendent edge* if x is the only neighbour of y in C .
- Let $y, y' \in V(C - \{v, x\})$ where $xy, xy', yy' \in E(C)$, and y, y' do not have any other neighbours in C . The subgraph $T = C[\{x, y, y'\}]$ is an *x-pendent triangle*.
- Let $z, y, y' \in V(C - \{v, x\})$ where $xy, xy', zy, zy', yy' \in E(C)$, and z, y, y' do not have any other neighbours in C . The subgraph $D = C[\{x, z, y, y'\}]$ is an *x-pendent diamond*.
- For some $t \geq 2$, let $z, y_1, \dots, y_t \in V(C - \{v, x\})$ where $xy_k, zy_k \in E(C)$ (resp. $xy_k, zy_k, xz \in E(C)$) for every $1 \leq k \leq t$, and z, y_1, \dots, y_t do not have any other neighbours in C . The subgraph $S = C[\{x, z, y_1, \dots, y_t\}]$ (resp. $S^+ = C[\{x, z, y_1, \dots, y_t\}]$) is an *x-pendent spindle* (resp. *x-pendent spindle⁺*).

Lemma 28. *Let $p \geq 2$, $\ell \geq 5$, $s \geq 0$. Let C be a connected $B_{\ell,s}$ -free graph, and $v, x \in V(C)$ where $d_C(v) = \Delta(C) \geq \ell + s - 1$. Let C' be a graph that can be obtained from C with any of the following operations.*

- (i) *Delete an x -pendent edge $e = xy$, and add the edge vy .*
- (ii) *Delete the three edges of an x -pendent triangle $T = C[\{x, y, y'\}]$, and add the edges vy, vy' .*
- (iii) *Delete the five edges of an x -pendent diamond $D = C[\{x, z, y, y'\}]$, and add the edges vz, vy, vy' .*
- (iv) *Delete the $2t$ edges of an x -pendent spindle $S = C[\{x, z, y_1, \dots, y_t\}]$ (for some $t \geq 2$), and add the edges vz, vy_1, \dots, vy_t .*
- (v) *Delete the $2t + 1$ edges of an x -pendent spindle⁺ $S^+ = C[\{x, z, y_1, \dots, y_t\}]$ (for some $t \geq 2$), and add the edges vz, vy_1, \dots, vy_t .*

Then C' is also $B_{\ell,s}$ -free, $d_{C'}(v) = \Delta(C') \geq \ell + s - 1$, and $e_p(C) < e_p(C')$.

Proof. Clearly we have $d_{C'}(v) = \Delta(C') \geq \ell + s - 1$, since in the transformation from C to C' , the only vertex whose degree has increased is v .

Next, let V_1 be the set of neighbours of v in C . Suppose that C' contains a copy of $B_{\ell,s}$, and we are in case (iv) or (v). Then for some $u_1, \dots, u_m \in \{z, y_1, \dots, y_t\}$ where $1 \leq m \leq t+1$, the edges vu_1, \dots, vu_m must be used by the $B_{\ell,s}$, with u_1, \dots, u_m being leaves. Note that $|V_1 \cup \{v, y_1, \dots, y_t, z\}| \geq \ell + s + t + 1$, and this means that there are vertices $w_1, \dots, w_m \in V_1$ which are not used in the $B_{\ell,s}$. Thus we obtain a copy of $B_{\ell,s}$ in C by replacing vu_1, \dots, vu_m with vw_1, \dots, vw_m , a contradiction. Similar arguments hold if we are in the other three cases, in view of $|V_1 \cup \{v, y\}| \geq \ell + s + 1$; $|V_1 \cup \{v, y, y'\}| \geq \ell + s + 2$; $|V_1 \cup \{v, z, y, y'\}| \geq \ell + s + 3$ for (i), (ii), (iii), respectively. Therefore, C' is $B_{\ell,s}$ -free.

It remains to prove that $e_p(C) < e_p(C')$ for each case.

(i) Going from C to C' , we see that the degree of v is increased by 1, and the degree of x is decreased by 1. Since $d_C(v) \geq d_C(x) \geq 2$, we have

$$\begin{aligned} e_p(C') - e_p(C) &= (d_C(v) + 1)^p - d_C(v)^p + (d_C(x) - 1)^p - d_C(x)^p \\ &\geq \sum_{1 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x)^{p-j}) + \binom{p}{2} (d_C(v)^{p-2} + d_C(x)^{p-2}) \\ &> 0. \end{aligned}$$

(ii) Going from C to C' , we see that the degree of v is increased by 2, the degree of x is decreased by 2, and the degrees of y, y' are each decreased from 2 to 1. Since $d_C(v) \geq$

$d_C(x) \geq 3$ and $d_C(v) \geq \ell + s - 1 \geq 4$, we have

$$\begin{aligned}
e_p(C') - e_p(C) &= (d_C(v) + 2)^p - d_C(v)^p + (d_C(x) - 2)^p - d_C(x)^p + 2(1^p - 2^p) \\
&\geq \sum_{1 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x)^{p-j}) 2^j \\
&\quad + \binom{p}{2} (d_C(v)^{p-2} + d_C(x)^{p-2}) 2^2 + 2(1 - 2^p) \\
&> 4(4^{p-2} + 3^{p-2}) - 2^{p+1} \geq 0.
\end{aligned}$$

(iii) Going from C to C' , we see that the degree of v is increased by 3, the degree of x is decreased by 2, the degree of z is decreased from 2 to 1, and the degrees of y, y' are each decreased from 3 to 1. Since $d_C(v) \geq d_C(x) \geq 3$ and $d_C(v) \geq \ell + s - 1 \geq 4$, we have

$$\begin{aligned}
e_p(C') - e_p(C) &= (d_C(v) + 3)^p - d_C(v)^p + (d_C(x) - 2)^p - d_C(x)^p \\
&\quad + (1^p - 2^p) + 2(1^p - 3^p) \\
&\geq p(3d_C(v)^{p-1} - 2d_C(x)^{p-1}) + \binom{p}{2} (9d_C(v)^{p-2} + 4d_C(x)^{p-2}) \\
&\quad + \sum_{3 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} 3^j - d_C(x)^{p-j} 2^j) + 3 - 2^p - 2 \cdot 3^p \\
&\geq 2 \cdot 4^{p-1} + 9 \cdot 4^{p-2} + 4 \cdot 3^{p-2} + 3 - 2^p - 2 \cdot 3^p > 0,
\end{aligned}$$

since $2 \cdot 4^{p-1} + 9 \cdot 4^{p-2} + 3 > 4^p + 3 > 2 \cdot 3^p$ and $4 \cdot 3^{p-2} \geq 2^p$.

(v) Going from C to C' , we see that the degree of v is increased by $t + 1$, the degree of x is decreased by $t + 1$, the degree of z is decreased from $t + 1$ to 1, and the degrees of y_1, \dots, y_t are each decreased from 2 to 1. Since $d_C(v) \geq d_C(x) \geq t + 1$, we have

$$\begin{aligned}
e_p(C') - e_p(C) &= (d_C(v) + t + 1)^p - d_C(v)^p + (d_C(x) - (t + 1))^p - d_C(x)^p \\
&\quad + 1^p - (t + 1)^p + t(1^p - 2^p) \\
&> \sum_{1 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x)^{p-j}) (t + 1)^j \\
&\quad + \binom{p}{2} (d_C(v)^{p-2} + d_C(x)^{p-2}) (t + 1)^2 - (t + 1)^p - t \cdot 2^p \\
&\geq 2(t + 1)^p - (t + 1)^p - t \cdot 2^p = (t + 1)^p - t \cdot 2^p > 0.
\end{aligned}$$

(iv) This follows from (v), since we can obtain the graph C'' from C by adding the edge xz , so that $e_p(C) < e_p(C'') < e_p(C')$. \square

Finally, the next lemma says that, in order to tackle Theorems 21 to 23 and Conjecture 24, it suffices to consider connected broom-free graphs with large maximum degree.

Lemma 29. *Suppose that the following statement is true: Let $p \geq 2$, $\ell \geq 6$, $s \geq 0$. Then there exists $d = d(\ell, s) \geq \ell + s$ such that, for all connected $B_{\ell, s}$ -free graph C with $c \geq d$ vertices and $\Delta(C) \geq d - 1$, we have $e_p(C) \leq e_p(H(c, \ell))$, with equality if and only if $C = H(c, \ell)$.*

Then Conjecture 24 holds for $n > (\ell + s + d)^2$.

Similarly, Theorem 21 holds if the following statement is true: Let $p \geq 2$ and $s \geq 0$. Then for all connected $B_{5,s}$ -free graph C with $c \geq s + 5$ vertices and $\Delta(C) \geq s + 4$, we have

$$e_p(C) \leq \begin{cases} e_p(H(c, 5)) & \text{if } s = 0, \\ e_p(K_1 + M_{c-1}) & \text{if } s \geq 1, \end{cases} \quad (6)$$

with equality if and only if $C = H(c, 5)$ for $s = 0$, and $C = K_1 + M_{c-1}$ for $s \geq 1$.

Proof. Suppose that the first assertion in Lemma 29 holds. Let $n > (\ell + s + d)^2$. Clearly the graph $H(n, \ell)$ is $B_{\ell,s}$ -free. Now, let G be a $B_{\ell,s}$ -free graph on n vertices and $G \neq H(n, \ell)$. Then the assertion of Conjecture 24 follows if we can prove that $e_p(G) < e_p(H(n, \ell))$.

Suppose first that $\Delta(G) \leq d - 2$. Then since $n - 1 > (d + 1)^2$, we have

$$\begin{aligned} (n - 1)^p &= (n - 1)(n - 1)^{p-1} > (n - 1)(d + 1)^{2p-2} \\ &> (n - 1)[d^{2p-2} + (2p - 2)d^{2p-3}] \\ &> (n - 1)d^{2p-2} + 2d^{2p-1} > n(d - 2)^p, \end{aligned}$$

so that

$$e_p(G) \leq n(d - 2)^p < (n - 1)^p < e_p(H(n, \ell)). \quad (7)$$

Now, let $\Delta(G) \geq d - 1$. Let $G^* \subset G$ be the subgraph consisting of the components with maximum degree at most $d - 2$, so that $\Delta(G^*) \leq d - 2$. We have $G = C_1 \cup \dots \cup C_t \cup G^*$ for some $t \geq 1$, where C_1, \dots, C_t are the components of G with maximum degree at least $d - 1$. Let $c_i = |V(C_i)| \geq d$. By the assertion in Lemma 29, for every $1 \leq i \leq t$, we have

$$e_p(C_i) \leq e_p(H(c_i, \ell)), \quad (8)$$

with equality if and only if $C_i = H(c_i, \ell)$. We apply (8) to every C_i , and then apply Lemma 26(b) repeatedly $t - 1$ times, and finally Lemma 27(b), if $|V(G^*)| > 0$. We find that $e_p(G) < e_p(H(n, \ell))$, since $G \neq H(n, \ell)$ by assumption.

By a similar argument, using Lemmas 26(a) and 27(a), and setting $\ell = 5$, $d = s + 5$, we see that the second assertion implies Theorem 21 for $n > (2s + 10)^2$. Note that the analogous inequalities to (7) would be

$$e_p(G) \leq n(s + 3)^p < (n - 1)^p < e_p(H(n, 5)) < e_p(K_1 + M_{n-1}).$$

□

We are now ready to prove Theorems 21, 22 and 23. The arguments in all three proofs are similar. In outline, it suffices to verify the statements in Lemma 29 for $\ell = 5, 6, 7$. Let C be a connected $B_{\ell,s}$ -free graph on c vertices as defined in the lemma. We may assume that C does not contain any of the pendent subgraphs, otherwise we may apply Lemma 28 to obtain another connected $B_{\ell,s}$ -free graph C' with $e_p(C) < e_p(C')$, so that we could consider the argument for C' instead of C . Under this assumption, we then show that $e_p(C) \leq e_p(K_1 + M_{c-1})$ for $\ell = 5$, $s \geq 1$, and $e_p(C) \leq e_p(H(c, \ell))$ otherwise. In each case, equality occurs if and only if C is the corresponding extremal graph.

Proof of Theorem 21. It suffices to verify the second statement in Lemma 29. Let C be a connected $B_{5,s}$ -free graph with $c \geq s + 5$ vertices, and $v \in V(C)$ with $d_C(v) = \Delta(C) \geq s + 4$. By Lemma 28, we may assume that C does not contain an x -pendent edge xy where $x, y \in V(C - v)$. Otherwise, we may delete xy and add vy to obtain the $B_{5,s}$ -free graph C' with $e_p(C) < e_p(C')$ and $d_{C'}(v) = \Delta(C')$, and consider the graph C' instead of C .

For $i \geq 1$, let V_i be the set of vertices of C at distance i from v . Note that $|V_1| = d_C(v) \geq s + 4$. Also, we have the following properties.

- (i) $V_i = \emptyset$ for $i \geq 3$.
- (ii) $C[V_2]$ does not contain an edge.
- (iii) Every vertex of V_2 has exactly one neighbour in V_1 .
- (iv) $C[V_1]$ contains at most one edge if $s = 0$, and $C[V_1]$ does not contain a copy of the path P_3 if $s \geq 1$.

Otherwise, suppose that (i) is false. Then we have a copy of $B_{5,s}$, where the path P_5 in $B_{5,s}$ is $x_3x_2x_1vy_1$ with $x_i \in V_i$ for $i = 1, 2, 3$, $y_1 \in V_1$, and the remaining s vertices of the $B_{5,s}$ are all in $V_1 \setminus \{x_1, y_1\}$. Properties (ii) to (iv) also hold for similar reasons. If $V_2 \neq \emptyset$, then we must have an edge $xy \in E(C)$ with $x \in V_1$ and $y \in V_2$. It follows from (i) to (iii) that xy is an x -pendent edge. Therefore, we may assume that $V_2 = \emptyset$. From (iv), we can now easily see that $C \subset H(c, 5)$ if $s = 0$; and $C \subset K_1 + M_{c-1}$ if $s \geq 1$, since $C[V_1]$ consists of independent edges and isolated vertices. Consequently (6) holds, as well as the cases of equality. \square

Proof of Theorem 22. It suffices to verify the first statement in Lemma 29 for $\ell = 6$, with $d = s + 6$. Let C be a connected $B_{6,s}$ -free graph with $c \geq s + 6$ vertices, and $v \in V(C)$ with $d_C(v) = \Delta(C) \geq s + 5$. We may assume that C does not contain an x -pendent edge or an x -pendent triangle, where $x \in V(C - v)$. Otherwise in either case, we may obtain the $B_{6,s}$ -free graph C' as described in Lemma 28 with $e_p(C) < e_p(C')$ and $d_{C'}(v) = \Delta(C')$, and consider the graph C' instead of C .

For $i \geq 1$, let V_i be the set of vertices of C at distance i from v . Note that $|V_1| = d_C(v) \geq s + 5$. Also, we have the following properties.

- (i) $V_i = \emptyset$ for $i \geq 4$.
- (ii) $C[V_i]$ does not contain a copy of the path P_{5-i} , for $i = 1, 2, 3$.
- (iii) Every vertex of V_3 has exactly one neighbour in V_2 .

Otherwise if any of (i) to (iii) is false, then we can easily find a copy of $B_{6,s}$ with centre v . By (i) to (iii), we may assume that $V_3 = \emptyset$, otherwise we have an x -pendent edge $xy \in E(C)$ where $x \in V_2$ and $y \in V_3$. Next, suppose that we have an edge yy' in $C[V_2]$. If y and y' have distinct neighbours in V_1 , then C again has a copy of $B_{6,s}$. Thus, y and y' must each have exactly one neighbour in V_1 , which is a common neighbour $x \in V_1$, and from (ii) with $i = 2$, we have $C[\{x, y, y'\}]$ is an x -pendent triangle. Therefore, we may assume that $C[V_2]$ does not contain an edge. Since no x -pendent edge xy exists where $x \in V_1$, $y \in V_2$, this means that every vertex of V_2 must have at least two neighbours in V_1 . Now, if $C[V_1 \cup V_2]$ contains

a copy of P_4 , say $y_1x_1x_2y_2$, then we must have $y_1, y_2 \in V_2$, otherwise C has a copy of $B_{6,s}$. Moreover, $x_1, x_2 \in V_1$, since $C[V_2]$ does not contain an edge. But then, y_1 has a neighbour in $V_1 \setminus \{x_1\}$, and we have another copy of P_4 with an end-vertex in V_1 , yielding a copy of $B_{6,s}$ in C . Therefore, $C[V_1 \cup V_2]$ does not contain a copy of P_4 , and we can easily deduce that $C[V_1 \cup V_2]$ is a subgraph whose components are stars and triangles.

Let C^* be the graph obtained from C by adding all edges from v to V_2 . Note that by replacing $C^* - v$ with the star of the same order, we obtain the graph $H(c, 6)$. We shall show that this operation does not decrease the value of e_p . Consider the following operations.

(A) Suppose that $C^* - v$ contains two star components, say with centres x and y , and the leaves at y are y_1, \dots, y_m for some $m \geq 0$. We delete the edges yy_1, \dots, yy_m and add the edges xy, xy_1, \dots, xy_m . The increase in the value of e_p is

$$(d_{C^*}(x) + m + 1)^p - d_{C^*}(x)^p + 2^p - (m + 1)^p > 2^p > 0.$$

(B) Suppose that $C^* - v$ contains at least two triangle components, say with vertices x_1, \dots, x_{3m} for some $m \geq 2$. We delete the $3m$ edges of the triangles, and connect x_1 to x_2, \dots, x_{3m} . The increase in the value of e_p is

$$(3m)^p - 3^p + (3m - 1)(2^p - 3^p) = (m^p - 3m)3^p + (3m - 1)2^p > 0.$$

(C) Suppose that $C^* - v$ contains a star and a triangle component, exactly one of each. Let x be the centre of the star, and note that since $|V(C^* - v)| = c - 1 \geq s + 5 \geq 5$, we have $d_{C^*}(x) \geq 2$. We delete the three edges of the triangle and connect x to its three vertices. The increase in the value of e_p is

$$(d_{C^*}(x) + 3)^p - d_{C^*}(x)^p + 3(2^p - 3^p).$$

If $p = 2$, then the increase is $6d_{C^*}(x) - 6 > 0$. If $p \geq 3$ and $d_{C^*}(x) = 2$, then the increase is $5^p + 2^{p+1} - 3^{p+1} > 0$. Otherwise if $p \geq 3$ and $d_{C^*}(x) \geq 3$, then the increase is at least

$$3pd_{C^*}(x)^{p-1} + 3(2^p - 3^p) \geq 3^{p+1} + 3(2^p - 3^p) > 0.$$

Therefore where possible, we apply operation (B), followed by successive applications of operation (A), and finally operation (C). We remark that during the applications of these operations, we may encounter a copy of $B_{6,s}$ in an intermediate graph. We obtain $e_p(C) \leq e_p(C^*) \leq e_p(H(c, 6))$. Equality occurs if and only if $C = C^*$ and $C^* - v$ is itself a star. That is, if and only if $C = H(c, 6)$. \square

Proof of Theorem 23. It suffices to verify the first statement in Lemma 29 for $\ell = 7$, with $d = 2s + 24$. Let C be a connected $B_{7,s}$ -free graph with $c \geq 2s + 24$ vertices, and $v \in V(C)$ with $d_C(v) = \Delta(C) \geq 2s + 23$. We may assume that C does not contain an x -pendent edge, triangle, diamond, spindle, or spindle⁺, where $x \in V(C - v)$. Otherwise, we may obtain the $B_{7,s}$ -free graph C' as described in Lemma 28 with $e_p(C) < e_p(C')$ and $d_{C'}(v) = \Delta(C')$, and consider the graph C' instead of C .

For $i \geq 1$, let V_i be the set of vertices of C at distance i from v . Note that $|V_1| = d_C(v) \geq 2s + 23$. Also, we have the following properties.

- (i) $V_i = \emptyset$ for $i \geq 5$.
- (ii) $C[V_i]$ does not contain a copy of the path P_{6-i} , for $i = 1, 2, 3, 4$.
- (iii) Every vertex of V_4 has exactly one neighbour in V_3 .

Otherwise if any of (i) to (iii) is false, then we can easily find a copy of $B_{7,s}$ with centre v . Proceeding exactly the same way as we did in Theorem 22, by avoiding a copy of $B_{7,s}$, or an x -pendent edge or triangle for some $x \in V(C - v)$, we can obtain the following facts.

- We may assume that $V_4 = \emptyset$.
- We may assume that $C[V_3]$ does not contain an edge, and that every vertex of V_3 has at least two neighbours in V_2 .

Suppose that we have $z \in V_3$. Then the only neighbours of z in C are $y_1, \dots, y_t \in V_2$, for some $t \geq 2$. In order to avoid a copy of $B_{7,s}$, note that each y_k must have exactly one neighbour in V_1 , which is a common neighbour $x \in V_1$. Also, no y_k can have a neighbour in $V_3 \setminus \{z\}$. Finally, no edge e in $C[V_2]$ can be incident with any y_k , except when $t = 2$ and $e = y_1y_2$. Thus, $C[\{x, z, y_1, \dots, y_t\}]$ is an x -pendent diamond or spindle. Therefore, we may further assume that $V_3 = \emptyset$.

By (ii) with $i = 2$, we see that the components of $C[V_2]$ are stars and triangles. Suppose that we have a star component in $C[V_2]$ with centre z and leaves y_1, \dots, y_t , for some $t \geq 2$. Then as before, the vertices y_k must each have exactly one neighbour in V_1 , which is a common neighbour $x \in V_1$. If z has a neighbour $x' \in V_1 \setminus \{x\}$, then we have a copy of $B_{7,s}$ with centre v , where the P_7 is $y_1xy_2zx'vx''$ for some $x'' \in V_1 \setminus \{x, x'\}$, and the s leaves are in $V_1 \setminus \{x, x', x''\}$. Therefore, x must be the unique neighbour of z in V_1 , and $C[\{x, z, y_1, \dots, y_t\}]$ is an x -pendent spindle⁺.

Thus, we may assume that the components of $C[V_2]$ are triangles, single edges, and isolated vertices. We consider the behaviour of the edges that connect these components to V_1 , keeping in mind that we should avoid creating a copy of $B_{7,s}$.

- If $y_1, y_2, y_3 \in V_2$ form a triangle in $C[V_2]$, then y_1, y_2, y_3 must have a unique common neighbour in V_1 , and they do not have any other neighbours in V_1 .
- Let y_1y_2 be a single edge in $C[V_2]$. If y_1, y_2 have exactly one common neighbour $x \in V_1$, then exactly one of y_1, y_2 has at least one neighbour in $V_1 \setminus \{x\}$, otherwise $C[\{x, y_1, y_2\}]$ is an x -pendent triangle or there is a copy of $B_{7,s}$. If y_1, y_2 have exactly two common neighbours $x_1, x_2 \in V_1$, then both y_1, y_2 cannot have a neighbour in $V_1 \setminus \{x_1, x_2\}$. Also, y_1, y_2 cannot have at least three common neighbours in V_1 . The remaining possibility is that y_1, y_2 have no common neighbour in V_1 .
- If y is an isolated vertex in $C[V_2]$, then y must have at least two neighbours in V_1 , otherwise there is an x -pendent edge xy , for some $x \in V_1$.

Let $\tilde{C} = (C - v) - E(C[V_1])$, i.e., \tilde{C} is the subgraph on $V_1 \cup V_2$, with the edges of $C[V_1]$ deleted. Then, when considering the components of \tilde{C} , these components are the subgraphs as shown in Figure 2(a). We refer the subgraphs illustrated as *Type 1* to *Type 5*.

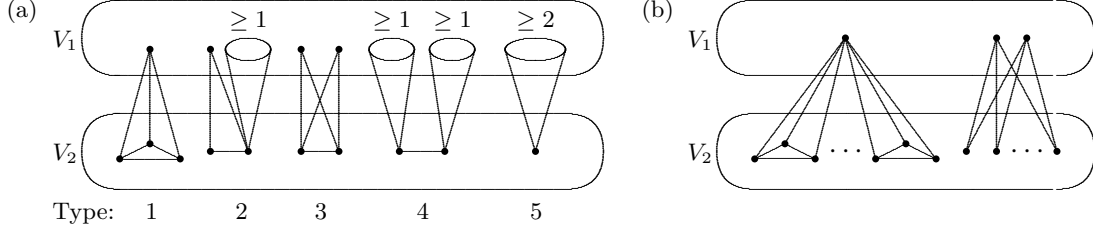


Figure 2. (a) Types 1 to 5 subgraphs; (b) How Type 1 and Type 5 subgraphs intersect

For two such subgraphs S, S' , we see that, in order to avoid creating a $B_{7,s}$ with centre v , we would need to have $V(S) \cap V(S') \cap V_1 = \emptyset$ in most cases. The only exceptions are when S and S' are of Type 1 and meeting at one vertex in V_1 , or they are of Type 5 with order 3 and meeting at exactly two vertices in V_1 . Indeed, we may have at least two subgraphs meeting in V_1 in these two exceptional cases, as shown in Figure 2(b). We next eliminate these two possibilities.

Suppose first that we have $x \in V_1$, with exactly $m \geq 2$ Type 1 subgraphs meeting at x . Let $y_1, \dots, y_{3m} \in V_2$ be the vertices of the triangles in V_2 . We delete the $6m$ edges of the Type 1 subgraphs, and add the edges vy_1, \dots, vy_{3m} . Then the degree of v is increased by $3m$, the degree of x is decreased by $3m$, and the degrees of the y_k are decreased from 3 to 1. Since $d_C(v) \geq d_C(x) > 3m$, the increase in the value of e_p is

$$\begin{aligned}
& (d_C(v) + 3m)^p - d_C(v)^p + (d_C(x) - 3m)^p - d_C(x)^p + 3m(1^p - 3^p) \\
> \sum_{1 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x)^{p-j})(3m)^j \\
& \quad + \binom{p}{2} (d_C(v)^{p-2} + d_C(x)^{p-2})(3m)^2 - 3m \cdot 3^p \\
> 18m^2(3m)^{p-2} - m \cdot 3^{p+1} = 2m^p \cdot 3^p - m \cdot 3^{p+1} \geq 4m \cdot 3^p - m \cdot 3^{p+1} > 0.
\end{aligned}$$

Next, suppose that we have exactly $m \geq 2$ Type 5 subgraphs of order 3 meeting at $x_1, x_2 \in V_1$. Let $y_1, \dots, y_m \in V_2$ be the vertices of these subgraphs in V_2 . Note that the neighbours of x_1 (resp. x_2) are precisely v and the y_k , and possibly x_2 (resp. x_1), otherwise there is a copy of $B_{7,s}$. Thus $d_C(x_1), d_C(x_2) \in \{m+1, m+2\}$. Suppose first that $m \leq s+1$. Note that $d_C(v) \geq 2s+23 > 2(m+2) \geq d_C(x_1) + d_C(x_2)$. We delete the $2m$ edges of the Type 5 subgraphs, and add the edges vy_1, \dots, vy_m . Then the degree of v is increased by m , the degrees of x_1, x_2 are decreased by m , and the degrees of the y_k are decreased from 2 to

1. The increase in the value of e_p is

$$\begin{aligned}
& (d_C(v) + m)^p - d_C(v)^p + (d_C(x_1) - m)^p - d_C(x_1)^p + (d_C(x_2) - m)^p \\
& \quad - d_C(x_2)^p + m(1^p - 2^p) \\
> & \sum_{1 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x_1)^{p-j} - d_C(x_2)^{p-j}) m^j \\
& \quad + \binom{p}{2} (d_C(v)^{p-2} + d_C(x_1)^{p-2} + d_C(x_2)^{p-2}) m^2 - m \cdot 2^p \\
> & ((2m)^{p-2} + 2m^{p-2}) m^2 - m \cdot 2^p = (2^{p-2} + 2) m^p - m \cdot 2^p > 0.
\end{aligned}$$

Secondly, let $m \geq s + 2$. Suppose that there is an edge $x'y'$ where either $x' \in V_1 \setminus \{x_1, x_2\}$ and $y' \in (V_1 \cup V_2) \setminus \{x_1, x_2, y_1, \dots, y_m\}$, or $x' = x_2$ and $y' \in V_1 \setminus \{x_1, x_2\}$. Then there is a copy of $B_{7,s}$ with centre x_1 , where the P_7 is $y'x'vx_2y_2x_1y_1$ or $wvy'x_2y_2x_1y_1$ for some $w \in V_1 \setminus \{x_1, x_2, y'\}$, and the s leaves are y_3, \dots, y_{s+2} . Similarly, we cannot have an edge x_1y' for every $y' \in V_1 \setminus \{x_1, x_2\}$. It follows that all the edges of C are those connecting v to V_1 , and all edges between $\{x_1, x_2\}$ and $\{y_1, \dots, y_m\}$, and possibly x_1x_2 . Now, let C' be the graph obtained by deleting the edges x_2y_1, \dots, x_2y_m and adding the edges vy_1, \dots, vy_m . Since $d_C(v) \geq d_C(x_2)$, the increase in the value of e_p is

$$\begin{aligned}
& (d_C(v) + m)^p - d_C(v)^p + (d_C(x_2) - m)^p - d_C(x_2)^p \\
\geq & \sum_{1 \leq j \leq p, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x_2)^{p-j}) m^j + \binom{p}{2} (d_C(v)^{p-2} + d_C(x_2)^{p-2}) m^2 > 0.
\end{aligned}$$

Moreover, we see that the degree sequence of C' is majorised by the degree sequence of $K_2 + E_{c-2}$, by identifying $\{v, x_1\}$ with K_2 , and the remaining vertices of C' with E_{c-2} . It follows that $e_p(C) < e_p(C') \leq e_p(K_2 + E_{c-2}) < e_p(H(c, 7))$.

Therefore, we may assume that no two of the subgraphs as shown in Figure 2(a) meet in V_1 . For such a subgraph S , let $[S]$ denote the component of $C - v$ containing S . We consider the structure of $[S]$, so as to avoid a copy of $B_{7,s}$. Clearly if S is of Type 1, then $[S] = S$. If S is of Type 2, 3 or 4, then either $[S] = S$, or $|V(S) \cap V_1| = 2$, and the edge connecting the two vertices of $V(S) \cap V_1$ is in $[S]$. Finally, let S be of Type 5, with $V(S) \cap V_1 = \{x_1, \dots, x_t\}$ for some $t \geq 2$. It is easy to check that any additional vertices and edges in $[S]$ are as follows. If $t = 2$, then we may possibly have the edge x_1x_2 , and either x_1 and x_2 are connected to a vertex of $V_1 \setminus \{x_1, x_2\}$, or only one of x_1, x_2 is connected to some vertices of $V_1 \setminus \{x_1, x_2\}$. If $t = 3$, then we may possibly have any number of the edges x_1x_2, x_1x_3, x_2x_3 , or none of these three edges and only one of x_1, x_2, x_3 is connected to some vertices of $V_1 \setminus \{x_1, x_2, x_3\}$. If $t \geq 4$, then we may either have exactly one edge in $\{x_1, \dots, x_t\}$, or no edge in $\{x_1, \dots, x_t\}$ and only one of x_1, \dots, x_t is connected to some vertices of $V_1 \setminus \{x_1, \dots, x_t\}$.

We see that all such components $[S]$ can be classified into exactly one of three types:

- (I) A subgraph of K_4 .
- (II) A $H(c', 5)$ for some $5 \leq c' \leq c - 1$ (i.e., a star on c' vertices with an edge connecting two leaves).

(III) A double star with at least five vertices (i.e., two disjoint stars with an edge connecting their centres). A star itself is a special case of a double star.

Moreover, by (ii) with $i = 1$, we see that if $Y = V_1 \setminus \bigcup V([S])$, where the union is taken over all such subgraphs S in Figure 2(a), then $C[Y]$ is P_5 -free. It is easy to show that the components of $C[Y]$ must also be one of the types (I), (II) or (III). Consequently, if we connect v to all vertices of V_2 to obtain the graph C^* , then the components of $C^* - v$ are of the types (I), (II) or (III). Note that by replacing $C^* - v$ with the graph $H(c-1, 5)$, we obtain the graph $H(c, 7)$. We shall show that this operation does not decrease the value of e_p . Consider the following operations.

(A) Suppose that $C^* - v$ has a double star component with at least five vertices, which is not a star. Let the centres be x, y , and the leaves at y be y_1, \dots, y_m , for some $m \geq 1$. We may assume that $d_{C^*}(x) \geq d_{C^*}(y) = m + 2$. We obtain the star with the same order by deleting the edges yy_1, \dots, yy_m , and adding the edges xy_1, \dots, xy_m . If $m \geq 2$, then the increase in the value of e_p is

$$\begin{aligned} (d_{C^*}(x) + m)^p - d_{C^*}(x)^p + 2^p - (m + 2)^p &> p d_{C^*}(x)^{p-1} m - (m + 2)^p \\ &\geq 2(m + 2)^{p-1} m - (m + 2)^p \geq 0. \end{aligned}$$

If $m = 1$, then we have $d_{C^*}(x) \geq 4$. In this case, the increase in the value of e_p is

$$(d_{C^*}(x) + 1)^p - d_{C^*}(x)^p + 2^p - 3^p > p d_{C^*}(x)^{p-1} + 2^p - 3^p \geq 2 \cdot 4^{p-1} + 2^p - 3^p > 0.$$

(B) Suppose that we have two components $C_1, C_2 \subset C^* - v$ with c_1 and c_2 vertices, where $c_1 \geq c_2 \geq 5$, and C_1 (resp. C_2) is either a star or the graph $H(c_1, 5)$ (resp. $H(c_2, 5)$). If C_1 is a star, we add an edge to create $H(c_1, 5)$, and likewise for C_2 , so that we have the graphs $H(c_1, 5)$ and $H(c_2, 5)$. We then delete all edges of the $H(c_2, 5)$, and connect all of its vertices to the universal vertex of the $H(c_1, 5)$, thus obtaining the subgraph $H(c_1 + c_2, 5)$. The increase in the value of e_p is at least

$$(c_1 + c_2)^p - c_1^p + 2^p - c_2^p + 2(2^p - 3^p) > p c_1^{p-1} c_2 - 2 \cdot 3^p > 0.$$

Let R be the subgraph of $C^* - v$ consisting of the components which are subgraphs of K_4 . We have $d_{C^*}(y) \leq 4$ for all $y \in V(R)$. Let $|V(R)| = r$.

(C) Suppose $r \geq 16$. We replace R with the star of order r , with centre $x \in V(R)$. The increase in the value of e_p is

$$r^p - d_{C^*}(x)^p + \sum_{y \in V(R-x)} (2^p - d_{C^*}(y)^p) > r^p - r \cdot 4^p \geq r(16^{p-1} - 4^p) \geq 0.$$

(D) Suppose that $1 \leq r \leq 15$, and the subgraph $C^* - (\{v\} \cup V(R))$ is $H(c_1, 5)$. Recall that $|V(C^* - v)| = c - 1 \geq 2s + 23 \geq 23$, and thus $c_1 \geq 8$. We delete all edges of R , and connect all vertices of R to the universal vertex of the $H(c_1, 5)$, to form a copy of $H(c-1, 5)$. Since $c_1 + r = c - 1$, the increase in the value of e_p is

$$\begin{aligned} (c-1)^p - c_1^p + \sum_{y \in V(R)} (2^p - d_{C^*}(y)^p) &\geq (c_1 + r)^p - c_1^p + r(2^p - 4^p) \\ &> p c_1^{p-1} r - r \cdot 4^p \geq r(2 \cdot 8^{p-1} - 4^p) \geq 0. \end{aligned}$$

Therefore where possible, we apply operation (C), then apply operation (A) to all double stars in V_1 , followed by successive applications of operation (B), and finally operation (D). Again, it is possible to encounter a copy $B_{7,s}$ in an intermediate graph during the applications of these operations. We obtain $e_p(C) \leq e_p(C^*) \leq e_p(H(c, 7))$. Equality occurs if and only if $C = C^*$ and $C^* - v$ is the graph $H(c - 1, 5)$. That is, if and only if $C = H(c, 7)$.

The proof of Theorem 23 is now complete. \square

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