

# Connected order ideals and $P$ -partitions

Ben P. Zhou\*

Center for Combinatorics, LPMC  
Nankai University  
Tianjin 300071, P.R. China  
benpzhou@163.com

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## Abstract

Given a finite poset  $P$ , we associate a simple graph denoted by  $G_P$  with all connected order ideals of  $P$  as vertices, and two vertices are adjacent if and only if they have nonempty intersection and are incomparable with respect to set inclusion. We establish a bijection between the set of maximum independent sets of  $G_P$  and the set of  $P$ -forests, introduced by Féray and Reiner in their study of the fundamental generating function  $F_P(\mathbf{x})$  associated with  $P$ -partitions. Based on this bijection, in the cases when  $P$  is naturally labeled we show that  $F_P(\mathbf{x})$  can factorise, such that each factor is a summation of rational functions determined by maximum independent sets of a connected component of  $G_P$ . This approach enables us to give an alternative proof for Féray and Reiner's nice formula of  $F_P(\mathbf{x})$  for the case of  $P$  being a naturally labeled forest with duplications. Another consequence of our result is a product formula to compute the number of linear extensions of  $P$ .

**Keywords:**  $P$ -partition;  $P$ -forest; linear extension; connected order ideal; maximum independent set

## 1 Introduction

Throughout this paper, we shall assume that  $P$  is a poset on  $\{1, 2, \dots, n\}$ . We use  $\leq_P$  to denote the order relation on  $P$  to distinguish from the natural order  $\leq$  on integers. We say that  $P$  is naturally labeled if  $i < j$  whenever  $i <_P j$ . A  $P$ -partition is a map  $f$  from  $P$  to the set  $\mathbb{N}$  of nonnegative integers such that

- (1) if  $i <_P j$ , then  $f(i) \geq f(j)$ ;
- (2) if  $i <_P j$  and  $i > j$ , then  $f(i) > f(j)$ .

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For more information on  $P$ -partitions, we refer the reader to the book [9] of Stanley or the recent survey paper [5] of Gessel. Let  $\mathcal{A}(P)$  denote the set of  $P$ -partitions. The fundamental generating function  $F_P(\mathbf{x})$  associated with  $P$ -partitions is defined as

$$F_P(\mathbf{x}) = \sum_{f \in \mathcal{A}(P)} \mathbf{x}^f = \sum_{f \in \mathcal{A}(P)} x_1^{f(1)} x_2^{f(2)} \cdots x_n^{f(n)}.$$

One of the most important problems in the theory of  $P$ -partitions is to determine explicit expressions for  $F_P(\mathbf{x})$ . The main objective of this paper is to show that for any naturally labeled finite poset  $P$ , the generating function  $F_P(\mathbf{x})$  can factorise.

Let us first review some background. The first explicit expression for  $F_P(\mathbf{x})$  was given by Stanley [8]. Recall that a linear extension of  $P$  is a permutation  $w = w_1 w_2 \cdots w_n$  on  $\{1, 2, \dots, n\}$  such that  $i < j$  whenever  $w_i <_P w_j$ . Let  $\mathcal{L}(P)$  be the set of linear extensions of  $P$ . For a permutation  $w$ , write

$$\text{Des}(w) = \{i \mid 1 \leq i \leq n-1, w_i > w_{i+1}\}$$

for the descent set of  $w$ . Stanley [8] showed that

$$F_P(\mathbf{x}) = \sum_{w \in \mathcal{L}(P)} \frac{\prod_{i \in \text{Des}(w)} x_{w_1} x_{w_2} \cdots x_{w_i}}{\prod_{j=1}^n (1 - x_{w_1} x_{w_2} \cdots x_{w_j})}. \quad (1)$$

Boussicault, Féray, Lascoux and Reiner [2] obtained a similar formula for  $F_P(\mathbf{x})$  when  $P$  is a forest, namely, every element of  $P$  is covered by at most one other element. We say that  $j$  is the parent of  $i$ , if  $i$  is covered by  $j$  in  $P$ . Björner and Wachs [1] defined the descent set of a forest  $P$  as

$$\text{Des}(P) = \{i \mid \text{if } j \text{ is the parent of } i, \text{ then } i > j\}. \quad (2)$$

Thus, if  $i \in \text{Des}(P)$ , then there exists a node  $j \in P$  such that  $i <_P j$  but  $i > j$ . In particular, when a forest  $P$  is naturally labeled, the descent set  $\text{Des}(P)$  is empty. For a forest  $P$ , Boussicault, Féray, Lascoux, and Reiner's formula is stated as

$$F_P(\mathbf{x}) = \frac{\prod_{i \in \text{Des}(P)} \prod_{k \leq_P i} x_k}{\prod_{j=1}^n \left(1 - \prod_{\ell \leq_P j} x_\ell\right)}. \quad (3)$$

Furthermore, Féray and Reiner [4] obtained a nice formula for  $F_P(\mathbf{x})$  when  $P$  is a naturally labeled forest with duplications, whose definition is given below. Recall that an order ideal of  $P$  is a subset  $J$  such that if  $i \in J$  and  $j \leq_P i$ , then  $j \in J$ . Throughout the rest of this paper, we will use  $J$  to represent an order ideal of  $P$ . An order ideal  $J$  is connected if the Hasse diagram of  $J$  is a connected graph. A poset  $P$  is called a forest with duplications if for any connected order ideal  $J_a$  of  $P$ , there exists at most one other connected order ideal  $J_b$  such that  $J_a$  and  $J_b$  intersect nontrivially, namely,

$$J_a \cap J_b \neq \emptyset, \quad J_a \not\subset J_b \quad \text{and} \quad J_b \not\subset J_a.$$

We would like to point out that a naturally labeled forest must be a naturally labeled forest with duplications, while the Hasse diagram of a naturally labeled forest with duplications needs not to be a forest. Let  $\mathcal{J}_{conn}(P)$  be the set of connected order ideals of  $P$ . For a naturally labeled forest with duplications, Féray and Reiner [4] proved that

$$F_P(\mathbf{x}) = \frac{\prod_{\{J_a, J_b\} \in \Pi(P)} \left(1 - \prod_{i \in J_a} x_i \prod_{j \in J_b} x_j\right)}{\prod_{J \in \mathcal{J}_{conn}(P)} \left(1 - \prod_{k \in J} x_k\right)}, \quad (4)$$

where  $\Pi(P)$  consists of all pairs  $\{J_a, J_b\}$  of connected order ideals that intersect nontrivially. Note that when  $P$  is a naturally labeled forest (with no duplication), both  $\text{Des}(P)$  and  $\Pi(P)$  are empty, and each connected order ideal  $J$  of  $P$  must equal to  $\{\ell \mid \ell \leq_P j\}$  for some  $j \in \{1, 2, \dots, n\}$  and vice versa, and hence formula (4) coincides with formula (3) in this special case.

For any poset  $P$ , Féray and Reiner [4] introduced the notion of  $P$ -forests and obtained a decomposition of the set  $\mathcal{L}(P)$  in terms of linear extensions of  $P$ -forests. Recall that a  $P$ -forest  $F$  is a forest on  $\{1, 2, \dots, n\}$  such that for any node  $i$ , the subtree rooted at  $i$  is a connected order ideal of  $P$ , and that for any two incomparable nodes  $i$  and  $j$  in the poset  $F$ , the union of the subtrees rooted at  $i$  and  $j$  is a disconnected order ideal of  $P$ . Let  $\mathcal{F}(P)$  stand for the set of  $P$ -forests. For example, for the poset  $P$  in Figure 1 there are three  $P$ -forests  $F_1, F_2$  and  $F_3$ .

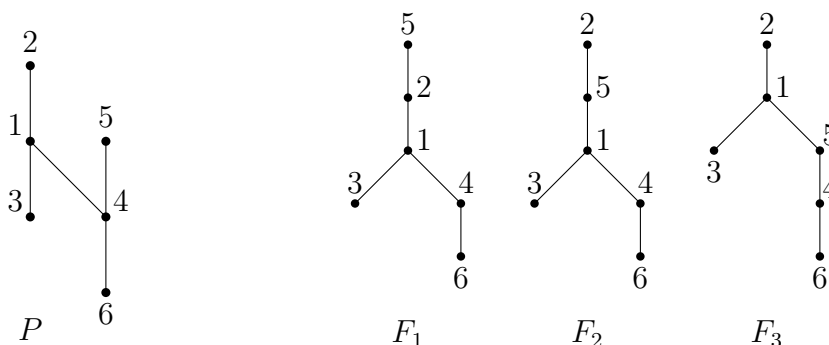


Figure 1: A poset  $P$  and the corresponding  $P$ -forests.

Féray and Reiner [4] showed that

$$\mathcal{L}(P) = \bigsqcup_{F \in \mathcal{F}(P)} \mathcal{L}(F), \quad (5)$$

which was implied in [4, Proposition 11.7]. As was remarked by Féray and Reiner, the decomposition in (5) also appeared in the work of Postnikov [6] and Postnikov, Reiner and Williams [7]. Combining (1), (3) and (5), one readily sees that

$$F_P(\mathbf{x}) = \sum_{F \in \mathcal{F}(P)} \frac{\prod_{i \in \text{Des}(F)} \prod_{k \leq_F i} x_k}{\prod_{j=1}^n \left(1 - \prod_{\ell \leq_F j} x_\ell\right)}. \quad (6)$$

Note that both (1) and (6) are summation formulas for  $F_P(\mathbf{x})$ . However, the expression of  $F_P(\mathbf{x})$  factored nicely for certain posets, as shown in (3) and (4). Thus it is desirable to ask that for more general posets  $P$  whether  $F_P(\mathbf{x})$  can factorise. In this paper, we show that  $F_P(\mathbf{x})$  can factorise for any naturally labeled poset  $P$ .

Before stating our result, let us first introduce some definitions and notations. In the following we always assume that  $P$  is a poset on  $\{1, 2, \dots, n\}$ . For any graph  $G$ , we use  $V(G)$  to denote the set of vertices of  $G$ . We associate to  $P$  a simple graph denoted by  $G_P$  with the set  $\mathcal{J}_{conn}(P)$  of connected order ideals of  $P$  as  $V(G_P)$ , and two vertices are adjacent if they intersect nontrivially. For example, if  $P$  is the poset given in Figure 1, then  $G_P$  is as illustrated in Figure 2, where we use  $\Lambda_i^P = \{k \mid k \leq_P i\}$  to denote the principal order ideal of  $P$  generated by  $i$ , and adopt the notation  $\Lambda_{i,j}^P = \Lambda_i^P \cup \Lambda_j^P$ .

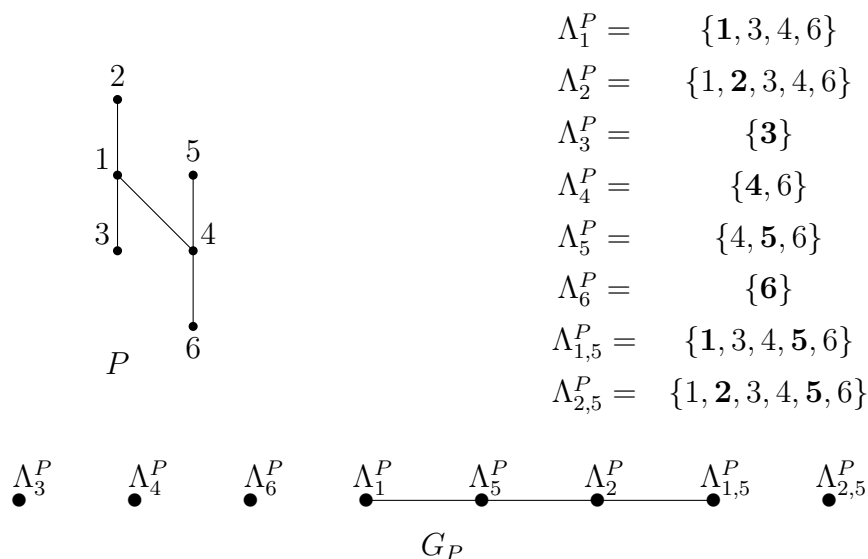


Figure 2: Connected order ideals of  $P$  and the graph  $G_P$ .

The first result of this paper is a bijection between the set of  $P$ -forests and the set of maximum independent sets of  $G_P$ . Recall that an independent set of a graph is a subset of vertices such that no two vertices of the subset are adjacent. A maximum independent set of a graph is an independent set that of largest possible size. For any graph  $G$ , we use  $\mathcal{M}(G)$  to denote the set of maximum independent sets of  $G$ . We have the following result.

**Theorem 1.** *There exists a bijection between the set  $\mathcal{F}(P)$  of  $P$ -forests and the set  $\mathcal{M}(G_P)$  of maximum independent sets of  $G_P$ .*

The proof of this result will be given in Section 2, where we establish a bijection  $\Phi$  from  $\mathcal{F}(P)$  to  $\mathcal{M}(G_P)$ . Let  $\Psi$  be the inverse map of  $\Phi$ . In view of the fact that  $\Psi(M)$  is a forest, for a maximum independent set  $M$  of  $G_P$ , we can define the descent set  $\text{Des}(M)$  of  $M$  as the descent set  $\text{Des}(\Psi(M))$ , namely,

$$\text{Des}(M) = \text{Des}(\Psi(M)), \tag{7}$$

where  $\text{Des}(\Psi(M))$  is given by (2). Suppose the graph  $G_P$  has  $h$  connected components, say  $C_1, C_2, \dots, C_h$ . As usual, we use  $V(C_r)$  to denote the vertex set of  $C_r$  for  $1 \leq r \leq h$ , respectively. It is clear that each maximum independent set of  $G_P$  is a disjoint union of maximum independent sets of  $G_P$ 's connected components. Let  $\mathcal{M}(C_r)$  denote the set of maximum independent sets of  $C_r$  for each  $1 \leq r \leq h$ , respectively. Given a  $M_r \in \mathcal{M}(C_r)$ , we shall further define a descent set for  $M_r$  as illustrated below. Let  $M$  be a maximum independent set of  $G_P$  such that  $M \cap V(C_r) = M_r$ . For any  $J \in M$ , let

$$\mu(M, J) = \bigcup_{J' \in M, J' \subset J} J'. \quad (8)$$

Define  $\text{Des}(M_r, M)$  and  $\overline{\text{Des}}(M_r, M)$  as

$$\begin{aligned} \text{Des}(M_r, M) &= \{i \in \text{Des}(M) \mid \{i\} = J \setminus \mu(M, J) \text{ for some } J \in M_r\}, \\ \overline{\text{Des}}(M_r, M) &= \{J \in M_r \mid J \setminus \mu(M, J) = \{i\} \text{ for some } i \in \text{Des}(M_r, M)\}. \end{aligned}$$

It is remarkable that  $\text{Des}(M_r, M)$  and  $\overline{\text{Des}}(M_r, M)$  are irrelevant to the choice of  $M$  when the poset  $P$  is naturally labeled. Precisely, we have the following result.

**Theorem 2.** *Suppose that  $P$  is a naturally labeled poset and  $G_P$  has connected components  $C_1, C_2, \dots, C_h$ . Let  $M_r$  be a maximum independent set of  $C_r$  for some  $1 \leq r \leq h$ . Then for any two maximum independent sets  $M^1, M^2$  of  $G_P$  satisfying  $M^1 \cap V(C_r) = M^2 \cap V(C_r) = M_r$ , we have*

$$\begin{aligned} \text{Des}(M_r, M^1) &= \text{Des}(M_r, M^2), \\ \overline{\text{Des}}(M_r, M^1) &= \overline{\text{Des}}(M_r, M^2). \end{aligned} \quad (9)$$

Therefore, for a naturally labeled poset  $P$  and a given  $M_r \in \mathcal{M}(C_r)$ , we can introduce the notation of  $\text{Des}(M_r)$  and  $\overline{\text{Des}}(M_r)$ , which are respectively defined by

$$\begin{aligned} \text{Des}(M_r) &= \text{Des}(M_r, M), \\ \overline{\text{Des}}(M_r) &= \overline{\text{Des}}(M_r, M), \end{aligned} \quad (10)$$

where  $M$  is some maximum independent set of  $G_P$  such that  $M \cap V(C_r) = M_r$ .

The main result of this paper is as follows.

**Theorem 3.** *If  $P$  is a naturally labeled poset, and the graph  $G_P$  has  $h$  connected components  $C_1, C_2, \dots, C_h$ . Then we have*

$$F_P(\mathbf{x}) = \prod_{r=1}^h \sum_{M_r \in \mathcal{M}(C_r)} \frac{\prod_{J \in \overline{\text{Des}}(M_r)} \prod_{k \in J} x_k}{\prod_{J \in M_r} (1 - \prod_{j \in J} x_j)}. \quad (11)$$

This paper is organized as follows. In Section 2, we shall give a proof of Theorem 1. In Section 3, we shall prove Theorems 2 and 3. Based on Theorem 3, we provide an alternative proof for Féray and Reiner's formula (4). In Section 4, Theorem 3 will be used to derive the generating function of major index of linear extensions of  $P$ , as well as to count the number of linear extensions of  $P$ .

## 2 The bijection $\Phi$ between $\mathcal{F}(P)$ and $\mathcal{M}(G_P)$

The aim of this section is to give a proof of Theorem 1. To this end, we shall establish a bijection  $\Phi$  from  $\mathcal{F}(P)$  to  $\mathcal{M}(G_P)$  as mentioned before.

To give a description of the map  $\Phi$ , we first note some properties of  $\mathcal{F}(P)$  and  $\mathcal{M}(G_P)$ . Given  $M \in \mathcal{M}(G_P)$  and  $J \in M$ , let

$$\begin{aligned} U(M, J) &= \{J' \in M \mid J' \subset J\}, \\ U_{max}(M, J) &= \{J_a \in U(M, J) \mid J_a \not\subset J_b \text{ for any } J_b \in U(M, J)\}. \end{aligned} \quad (12)$$

Recall that the set  $\mu(M, J)$  is defined in (8), which is also an order ideal of  $P$ . Thus

$$\mu(M, J) = \bigcup_{J' \in U(M, J)} J' = \bigcup_{J' \in U_{max}(M, J)} J'. \quad (13)$$

The following assertion will be used in the future proofs.

**Lemma 4.** *For any  $M \in \mathcal{M}(G_P)$  and  $J \in M$ , the intersection of any two elements of  $U_{max}(M, J)$  is empty.*

*Proof.* Let  $J_1, J_2 \in U_{max}(M, J)$ . Because  $U_{max}(M, J) \subset M$  and  $M$  is an independent set of  $G_P$ , it follows that  $J_1$  and  $J_2$  are not adjacent in  $G_P$ . Recall that for any two vertices  $J_1, J_2 \in \mathcal{J}_{conn}(P)$  of  $G_P$ ,  $J_1$  and  $J_2$  are not adjacent in  $G_P$  if and only if

$$J_1 \cap J_2 = \emptyset, \quad \text{or } J_1 \subset J_2, \quad \text{or } J_2 \subset J_1.$$

On the other hand, by the definition of  $U_{max}(M, J)$ , there is neither  $J_1 \subset J_2$  nor  $J_2 \subset J_1$ . Hence  $J_1 \cap J_2 = \emptyset$ .  $\square$

Given a  $P$ -forest  $F \in \mathcal{F}(P)$ , let  $\Lambda_i^F = \{j \mid j \leq_F i\}$  denote the principal order ideal of  $F$  generated by  $i$ . By definition of  $P$ -forest, each  $\Lambda_i^F$  is a connected order ideal of  $P$ , although  $\Lambda_i^F$  is not necessarily a principal order ideal of  $P$ . Then by the definition of  $G_P$ , each  $\Lambda_i^F$  is a vertex of  $G_P$ . Moreover, we have the following result.

**Lemma 5.** *For any  $P$ -forest  $F \in \mathcal{F}(P)$ , the principal order ideals  $\Lambda_1^F, \Lambda_2^F, \dots, \Lambda_n^F$  form a maximum independent set of  $G_P$ .*

*Proof.* We first show that  $\{\Lambda_1^F, \Lambda_2^F, \dots, \Lambda_n^F\}$  is an independent set of  $G_P$ , that is, for any two nodes  $i, j$  of  $F$ , the principal order ideals  $\Lambda_i^F$  and  $\Lambda_j^F$  are not adjacent in  $G_P$ . There are two cases to consider.

- (1) The vertices  $i$  and  $j$  are incomparable in  $F$ . Since  $F$  is a forest, it is clear that  $\Lambda_i^F \cap \Lambda_j^F = \emptyset$ . This implies that  $\Lambda_i^F$  and  $\Lambda_j^F$  are not adjacent in  $G_P$ .
- (2) The vertices  $i$  and  $j$  are comparable in  $F$ . If  $i <_F j$ , then  $\Lambda_i^F \subset \Lambda_j^F$ ; If  $j <_F i$ , then  $\Lambda_j^F \subset \Lambda_i^F$ . In both circumstances,  $\Lambda_i^F$  and  $\Lambda_j^F$  are not adjacent in  $G_P$ .

We proceed to show that the independent set  $\{\Lambda_1^F, \Lambda_2^F, \dots, \Lambda_n^F\}$  is of the largest possible size. To this end, it is enough to verify that  $|M| \leq n$  for any independent set  $M$  of  $G_P$ . Assume that  $M = \{J_1, J_2, \dots, J_k\}$  is an independent set of  $G_P$ , which means that  $J_i$  is a connected order ideal of  $P$ , and  $J_i, J_j$  are not adjacent in  $G_P$  for any  $1 \leq i < j \leq k$ . We further assume that the subscript satisfies  $r < s$  whenever  $J_r \subset J_s$ . In fact, this can be achieved as follows. Consider  $M$  as a poset ordered by set inclusion. Then choose a subscript such that  $J_1 J_2 \cdots J_k$  is a linear extension of  $M$ . Such a subscript satisfies the condition that  $r < s$  whenever  $J_r \subset J_s$ .

For  $1 \leq s \leq k$ , let

$$I_s = \bigcup_{1 \leq r \leq s} J_r.$$

It is clear that  $I_{s-1} \subset I_s$  for any  $1 < s \leq k$ . We claim that

$$\emptyset \neq I_1 \subset I_2 \subset \cdots \subset I_k \subseteq \{1, 2, \dots, n\}, \quad (14)$$

which implies that  $|M| = k \leq n$ .

Suppose to the contrary that  $I_s = I_{s-1}$  for some  $1 < s \leq k$ . Thus,

$$J_s \subseteq I_s = I_{s-1} = \bigcup_{1 \leq r \leq s-1} J_r. \quad (15)$$

The set  $U(M, J_s)$  is defined as

$$U(M, J_s) = \{J' \mid J' \in M, J' \subset J_s\} = \{J_r \mid 1 \leq r \leq s-1, J_r \subset J_s\}.$$

Clearly,

$$\mu(M, J_s) = \bigcup_{J' \in U(M, J_s)} J'. \quad (16)$$

Notice that for any  $1 \leq r \leq s-1$ , if  $J_r$  does not belong to  $U(M, J_s)$ , then  $J_r \cap J_s = \emptyset$ , since otherwise  $J_r$  and  $J_s$  intersect nontrivially, contradicting the assumption that  $M$  is an independent set of  $G_P$ . In view of relation (15), we have

$$J_s \subseteq \bigcup_{J' \in U(M, J_s)} J' = \mu(M, J_s),$$

which together with (13) and (16), leads to

$$J_s = \mu(M, J_s) = \bigcup_{J' \in U_{max}(M, J_s)} J'.$$

If  $U_{max}(M, J_s)$  has only one element, say,  $U_{max}(M, J_s) = \{J_r\}$  for some  $1 \leq r \leq s-1$ , then  $J_s = J_r$ , which is contrary to  $J_r \subset J_s$ . Next we may assume that  $U_{max}(M, J_s)$  has more than one element. By Lemma 4, the intersection of any two elements of  $U_{max}(M, J_s)$  is empty. Thus  $J_s$  is the union of some (at least two) nonintersecting connected order ideals, which can not be connected. This contradicts the fact that  $J_s$  is a connected order ideal. It follows that  $I_{s-1} \subset I_s$  for each  $1 < s \leq k$ , as desired.  $\square$

By the above lemma, we can define a map  $\Phi : \mathcal{F}(P) \longrightarrow \mathcal{M}(G_P)$  by letting

$$\Phi(F) = \{\Lambda_1^F, \Lambda_2^F, \dots, \Lambda_n^F\}$$

for any  $F \in \mathcal{F}(P)$ . In order to show that  $\Phi$  is a bijection, we shall construct the inverse map of  $\Phi$ , denoted by  $\Psi$ . To give a description of  $\Psi$ , we need the following lemma.

**Lemma 6.** *Given  $M \in \mathcal{M}(G_P)$  and  $J \in M$ , there exists a unique  $j$  such that*

$$J \setminus \mu(M, J) = \{j\}, \tag{17}$$

where  $\mu(M, J)$  is given in (8). Moreover,  $j$  is a maximal element of  $J$  with respect to the order  $\leq_P$ , and

$$J_r \setminus \mu(M, J_r) \neq J_s \setminus \mu(M, J_s) \tag{18}$$

for any distinct  $J_r, J_s \in M$ .

*Proof.* By Lemma 5, we see that each maximum independent set of  $G_P$  should contain  $n$  vertices. Suppose that  $M = \{J_1, J_2, \dots, J_n\}$ . As in the proof of Lemma 5, we may assume that

$$r < s \text{ whenever } J_r \subset J_s. \tag{19}$$

For  $1 \leq s \leq n$ , let

$$I_s = \bigcup_{1 \leq r \leq s} J_r.$$

By (14), we see that

$$\emptyset \neq I_1 \subset I_2 \subset \dots \subset I_n \subseteq \{1, 2, \dots, n\}. \tag{20}$$

Therefore, if setting  $I_0 = \emptyset$ , we obtain that for  $1 \leq s \leq n$ ,

$$|I_s \setminus I_{s-1}| = 1. \tag{21}$$

Let  $J = J_s$  for some  $1 \leq s \leq n$ . In view of (8) and (19), we get that

$$\mu(M, J_s) = \bigcup_{J' \in M, J' \subset J_s} J' = \bigcup_{1 \leq r \leq s-1, J_r \subset J_s} J_r \subseteq I_{s-1}.$$

Thus we have

$$J \setminus \mu(M, J) = J_s \setminus \mu(M, J_s) = J_s \setminus I_{s-1} = I_s \setminus I_{s-1}, \tag{22}$$

where the second equality follows from the fact that for any  $1 \leq r \leq s-1$ , either  $J_r \subset J_s$  or  $J_r \cap J_s = \emptyset$ . In view of (21) and (22), we arrive at (17) and (18).

It remains to show that the unique element  $j$  of  $J_s \setminus \mu(M, J_s)$  is a maximal element of  $J_s$  with respect to the order  $\leq_P$ . Suppose that  $j$  is not a maximal element of  $J_s$ . Then there exists a maximal element  $i$  of  $J_s$  such that  $j <_P i$ . By (17) and  $j \neq i$ , we see that  $i \in \mu(M, J_s)$ . Therefore, there exists some  $J' \subset J_s$  of and  $J' \in M$  such that  $i \in J'$ . Since  $J'$  is an order ideal of  $P$ , we get  $j \in J' \subseteq \mu(M, J_s)$ , contradicting with the fact  $j \notin \mu(M, J_s)$ .  $\square$



For any  $M \in \mathcal{M}(G_P)$ , it follows from (17) and (18) that

$$\{1, 2, \dots, n\} = \bigsqcup_{J \in M} J \setminus \mu(M, J).$$

Let  $F_M$  be the poset on  $\{1, 2, \dots, n\}$  such that  $i <_{F_M} j$  if and only if  $J_a \subset J_b$ , where  $J_a$  and  $J_b$  are the two connected order ideals in  $M$  satisfies  $J_a \setminus \mu(M, J_a) = \{i\}$ ,  $J_b \setminus \mu(M, J_b) = \{j\}$ . The following result show an important property for principal order ideals of the poset  $F_M$ .

**Lemma 7.** *Given  $M \in \mathcal{M}(G_P)$ , let  $F_M$  be the poset defined as above. Then for any  $1 \leq j \leq n$  we have  $\Lambda_j^{F_M} = \{i \mid i \leq_{F_M} j\} = J$ , where  $J \in M$  satisfying  $J \setminus \mu(M, J) = \{j\}$  as in Lemma 6.*

*Proof.* We use the principle of Noetherian induction.

If  $j$  is a minimal element of  $F_M$  with respect to the order  $\leq_{F_M}$ , then  $J$  is also a minimal element of  $M$  when  $M$  is regarded as a poset ordered by set inclusion. Hence  $\Lambda_j^{F_M} = \{j\}$  and there exists no  $J' \in M$  such that  $J' \subset J$ , which yields that  $\mu(M, J) = \emptyset$ . So  $J = \{j\} \cup \mu(M, J) = \{j\}$ , and then  $\Lambda_j^{F_M} = J$ .

Suppose that  $j$  is not a minimal element of  $F_M$  (with respect to the order  $\leq_{F_M}$ ) and  $\Lambda_i^{F_M} = J'$  holds for any  $i <_{F_M} j$ , where  $J' \setminus \mu(M, J') = \{i\}$ . The construction of  $F_M$  tells us that  $i <_{F_M} j$  if and only if  $J' \subset J$ . Since  $\Lambda_i^{F_M} \subset \Lambda_j^{F_M}$  holds for each  $i <_{F_M} j$ , we have

$$\Lambda_j^{F_M} = \{i \mid i \leq_{F_M} j\} = \{j\} \cup \left( \bigcup_{i <_{F_M} j} \Lambda_i^{F_M} \right).$$

Then by the induction hypothesis, we get that

$$\Lambda_j^{F_M} = \{j\} \cup \left( \bigcup_{J' \in M, J' \subset J} J' \right) = \{j\} \cup \mu(M, J) = J. \quad \square$$

We proceed to examine more structure of  $F_M$ , and obtain the following result.

**Lemma 8.** *For any  $M \in \mathcal{M}(G_P)$ , the poset  $F_M$  is a  $P$ -forest.*

*Proof.* We first show that  $F_M$  is a forest. Suppose otherwise that  $F_M$  is not a forest. Then there exists an element  $i$  in  $F_M$  such that  $i$  is covered by at least two elements of  $F_M$ , say  $j, k$ . Thus  $j$  and  $k$  must be incomparable with respect to the order  $\leq_{F_M}$ . (Recall that in a poset  $P$ , we say that an element  $u$  is covered by an element  $v$  if  $u <_P v$  and there is no element  $w$  such that  $u <_P w <_P v$ .) By Lemma 6, there exist  $J_a, J_b, J_c \in M$  such that  $J_a \setminus \mu(M, J_a) = \{i\}$ ,  $J_b \setminus \mu(M, J_b) = \{j\}$  and  $J_c \setminus \mu(M, J_c) = \{k\}$ . By the construction of  $F_M$ , we see that  $J_a \subset J_b$ ,  $J_a \subset J_c$  and  $J_b, J_c$  are incomparable in  $M$  with respect to the set inclusion order. Hence,  $J_b \not\subset J_c$ ,  $J_c \not\subset J_b$  and  $(J_b \cap J_c) \supseteq J_a \neq \emptyset$ . This implies that  $J_b$  and  $J_c$  are adjacent in the graph  $G_P$ , contradicting the fact that  $M$  is an independent set.

We proceed to show that  $F_M$  is a  $P$ -forest. By Lemma 7, for each element  $i$  of  $F_M$ , the subtree  $\Lambda_i^{F_M} = \{j \mid j \leq_{F_M} i\}$  of  $F_M$  rooted at  $i$  is a connected order ideal of  $P$ . To verify that  $F_M$  is a  $P$ -forest, we still need to check that for  $1 \leq i, j \leq n$ , if  $i$  and  $j$  are incomparable in  $F_M$ , then the union  $\Lambda_i^{F_M} \cup \Lambda_j^{F_M}$  is a disconnected order ideal of  $P$ . By Lemma 6, assume that  $J_a$  and  $J_b$  are the connected order ideals in  $M$  such that  $J_a \setminus \mu(M, J_a) = \{i\}$  and  $J_b \setminus \mu(M, J_b) = \{j\}$ . By Lemma 7, we have  $J_a = \Lambda_i^{F_M}$  and  $J_b = \Lambda_j^{F_M}$ . Since  $i$  and  $j$  are incomparable in  $F_M$ , we obtain that  $J_a \not\subset J_b$  and  $J_b \not\subset J_a$ . On the other hand,  $J_a$  and  $J_b$  are not adjacent in the graph  $G_P$ . This allows us to conclude that  $J_a \cap J_b = \emptyset$ . Therefore, as an order ideal of  $P$ , the union  $J_a \cup J_b$  is disconnected, so is the union  $\Lambda_i^{F_M} \cup \Lambda_j^{F_M}$ . Hence  $F_M$  is a  $P$ -forest.  $\square$

With the above lemma, we can define the inverse map of  $\Phi$ , denoted by  $\Psi : \mathcal{M}(G_P) \rightarrow \mathcal{F}(P)$ , by letting

$$\Psi(M) = F_M$$

for any  $M \in \mathcal{M}(G_P)$ .

Now we are in a position to give a proof of Theorem 1.

*Proof of Theorem 1.* We first prove that  $\Psi(\Phi(F)) = F$  for any  $P$ -forest  $F$  and  $\Phi(\Psi(M)) = M$  for any maximum independent set  $M$  of  $G_P$ . The proof of the former statement will be given below, and the proof of the latter will be omitted here. Given a  $P$ -forest  $F$ , by definition, the image of  $F$  under the map  $\Phi$  is  $\Phi(F) = \{\Lambda_1^F, \dots, \Lambda_n^F\}$ , which is a maximum independent set of  $G_P$  by Lemma 5. Of course, we have  $\Lambda_i^F \subset \Lambda_j^F$  if and only if  $i <_F j$ . For each  $1 \leq i \leq n$  let  $J_i = \Lambda_i^F$  and then denote  $M = \{J_1, J_2, \dots, J_n\}$ . We proceed to show that  $\Psi(M) = F_M = F$ . Note that both  $F_M$  and  $F$  are posets on  $\{1, 2, \dots, n\}$ . It remains to show that  $i <_{F_M} j$  if and only if  $i <_F j$  for any  $i, j \in \{1, 2, \dots, n\}$ . Recall that for  $1 \leq i \leq n$  the principal order ideal  $\Lambda_i^F$  is the subtree of  $F$  rooted at  $i$ . Hence

$$J_i \setminus \mu(M, J_i) = \Lambda_i^F \setminus \left( \bigcup_{j <_F i} \Lambda_j^F \right) = \{i\}$$

holds for each  $1 \leq i \leq n$ . By the construction of  $F_M$ , we know that  $i <_{F_M} j$  if and only if  $J_i \subset J_j$ . On the other hand, in the given  $P$ -forest  $F$ ,  $i <_F j$  if and only if  $\Lambda_i^F \subset \Lambda_j^F$ . Since  $J_i = \Lambda_i^F$  for each  $1 \leq i \leq n$ , it follows that  $i <_{F_M} j$  if and only if  $i <_F j$ . Thus  $F_M = F$ , as desired.

Because  $\Psi(\Phi(F)) = F$  for any  $P$ -forest  $F$ , the map  $\Phi$  is one-to-one. Moreover, since the map  $\Psi$  is applicable to any maximum independent set  $M$  of  $G_P$ , the quality  $\Phi(\Psi(M)) = M$  ensures that  $\Phi$  is onto. Then  $\Phi$  is bijective.  $\square$

We take the poset  $P$  in Figure 1 as an example to illustrate Theorem 1 and its proof. There are there  $P$ -forests  $F_1, F_2$  and  $F_3$  as shown in Figure 1. The graph  $G_P$ , as shown in Figure 2, has three maximum independent sets:

$$\begin{aligned} M^1 &= \{\Lambda_3^P, \Lambda_4^P, \Lambda_6^P, \Lambda_1^P, \Lambda_2^P, \Lambda_{2,5}^P\}, \\ M^2 &= \{\Lambda_3^P, \Lambda_4^P, \Lambda_6^P, \Lambda_1^P, \Lambda_{1,5}^P, \Lambda_{2,5}^P\}, \\ M^3 &= \{\Lambda_3^P, \Lambda_4^P, \Lambda_6^P, \Lambda_5^P, \Lambda_{1,5}^P, \Lambda_{2,5}^P\}. \end{aligned}$$

The principal order ideals of  $F_1$  is as shown in Figure 3.

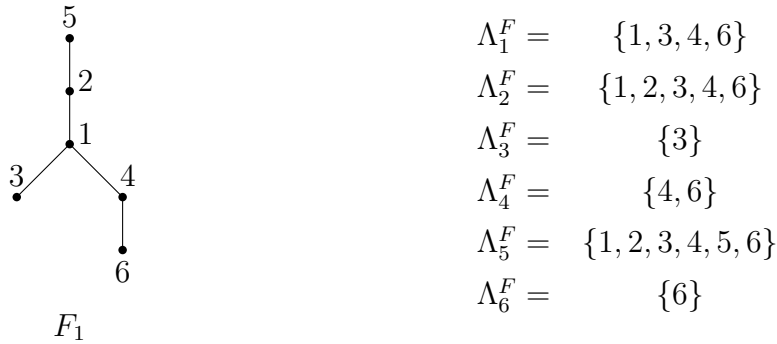


Figure 3: The  $P$ -forest  $F_1$  and its principal order ideals.

By the construction of  $\Phi$ , we have

$$\begin{aligned} \Phi(F_1) &= \{\Lambda_1^{F_1}, \Lambda_2^{F_1}, \dots, \Lambda_6^{F_1}\} \\ &= \{\{1, 3, 4, 6\}, \{1, 2, 3, 4, 6\}, \{3\}, \{4, 6\}, \{1, 2, 3, 4, 5, 6\}, \{6\}\}, \end{aligned}$$

which coincides with  $M^1$ . One can also verify that  $\Phi(F_2) = M^2$  and  $\Phi(F_3) = M^3$ .

On the other hand, for the maximum independent set  $M^1$ , if we set  $J_1 = \Lambda_1^P = \{1, 3, 4, 6\}$ ,  $J_2 = \Lambda_2^P = \{1, 2, 3, 4, 6\}$ ,  $J_3 = \Lambda_3^P = \{3\}$ ,  $J_4 = \Lambda_4^P = \{4, 6\}$ ,  $J_5 = \Lambda_{2,5}^P = \{1, 2, 3, 4, 5, 6\}$ ,  $J_6 = \Lambda_6^P = \{6\}$ , then it is straightforward to verify that  $J_i \setminus \mu(M^1, J_i) = \{i\}$  for  $1 \leq i \leq 6$ . And then, by definition, in the  $P$ -forest  $F_{M^1}$  there is  $2 <_{F_{M^1}} 5$ ,  $1 <_{F_{M^1}} 2$ ,  $3 <_{F_{M^1}} 1$ ,  $4 <_{F_{M^1}} 1$ ,  $6 <_{F_{M^1}} 4$ . One readily sees that  $F_{M^1} = F_1$ . Similarly, one can verify that  $F_{M^2} = F_2$  and  $F_{M^3} = F_3$ .

### 3 $F_P(\mathbf{x})$ for naturally labeled $P$

The main objective of this section is to prove Theorems 2 and 3. The proofs are based on some properties of certain subgraphs of  $G_P$ . Although we require that the poset  $P$  in Theorems 2 and 3 be naturally labeled, these properties of  $G_P$  are valid for any finite poset  $P$ .

To begin with, let us first introduce some notations. For an order ideal  $J$  of  $P$ , let  $gs(J)$  denote the set of maximal elements of  $J$  with respect to the order  $\leq_P$ , namely,

$$gs(J) = \{i \in J \mid \text{there exists no } j \in J \text{ such that } i <_P j\}.$$

This set is also called the generating set of  $J$ . Clearly, when  $gs(J) = \{i_1, i_2, \dots, i_k\}$ , we have  $J = \Lambda_{i_1}^P \cup \Lambda_{i_2}^P \cup \dots \cup \Lambda_{i_k}^P$ . Let  $\chi_J$  be the subgraph of  $G_P$  induced by the vertex subset  $\{\Lambda_{i_1}^P, \Lambda_{i_2}^P, \dots, \Lambda_{i_k}^P\}$ . We have the following assertion.

**Lemma 9.** *For any connected order ideal  $J$  of  $P$ , the graph  $\chi_J$  is connected.*

*Proof.* Assume that  $gs(J) = \{i_1, i_2, \dots, i_k\}$ . The proof is immediate if  $k = 1$ . In the following we shall assume that  $k \geq 2$ . Define

$$\text{Conn}(i_1) = \{i_r \in gs(J) \mid \text{there is a path in } \chi_J \text{ connecting } \Lambda_{i_1}^P \text{ and } \Lambda_{i_r}^P\}.$$

Note that  $i_1$  is always contained in  $\text{Conn}(i_1)$ . It is enough to show that  $\text{Conn}(i_1) = gs(J)$ . Otherwise, suppose that  $\text{Conn}(i_1) \neq gs(J)$ . Let

$$I_1 = \bigcup_{j \in \text{Conn}(i_1)} \Lambda_j^P \quad \text{and} \quad I_2 = \bigcup_{j \in gs(J) \setminus \text{Conn}(i_1)} \Lambda_j^P.$$

Then both  $I_1$  and  $I_2$  are nonempty subsets of  $J$  satisfying that  $I_1 \cup I_2 = J$ , and both  $I_1$  and  $I_2$  are order ideals of  $P$ . Since  $J$  is a connected order ideal of  $P$ , it follows that  $I_1 \cap I_2 \neq \emptyset$ . Thus there exists some  $u \in \text{Conn}(i_1)$  and some  $v \in gs(J) \setminus \text{Conn}(i_1)$  such that  $\Lambda_u^P \cap \Lambda_v^P \neq \emptyset$ . Since both  $u$  and  $v$  are maximal elements in the connected order ideal  $J$ , we must have  $\Lambda_u^P \not\subset \Lambda_v^P$  and  $\Lambda_v^P \not\subset \Lambda_u^P$ . This means that  $\Lambda_u^P$  and  $\Lambda_v^P$  are adjacent, implying that  $v \in \text{Conn}(i_1)$ . This leads to a contradiction.  $\square$

We also need the following lemma.

**Lemma 10.** *Let  $J$  be a connected order ideal of  $P$ , and let  $C$  be any connected subgraph of  $G_P$ . Assume that  $J$  is not adjacent to any vertex of  $C$ . If there exists a vertex  $J_a$  of  $C$  such that  $J_a \subset J$ , then  $J_b \subset J$  for any vertex  $J_b$  of  $C$ .*

*Proof.* We first consider the case when  $J_a$  and  $J_b$  are adjacent. In this case,  $J_b$  and  $J_a$  intersect nontrivially, and so we have  $\emptyset \neq (J_a \cap J_b)$ . On the other hand, since  $J_a \subset J$ , we obtain that

$$\emptyset \neq (J_a \cap J_b) \subset (J \cap J_b). \tag{23}$$

Combining (23) and the hypothesis that the vertices  $J_b$  and  $J$  are not adjacent, we get that  $J_b \subset J$  or  $J \subset J_b$ . If  $J \subset J_b$ , then  $J_a \subset J \subset J_b$ , which is impossible because  $J_a$  and  $J_b$  intersect nontrivially. Hence we have  $J_b \subset J$ .

We now consider the case when  $J_a$  is not adjacent to  $J_b$ . Since  $C$  is connected, there exists a sequence  $(J_0 = J_a, J_1, \dots, J_k = J_b)$  ( $k \geq 2$ ) of vertices of  $C$  such that  $J_i$  is adjacent to  $J_{i-1}$  for  $1 \leq i \leq k$ . By the above argument,  $J_1$  is contained in  $J$ . Therefore, by a simple recursion we get that  $J_b \subset J$ .  $\square$

For example, let  $P$  be the poset given in Figure 4. The graph  $G_P$  is illustrated in Figure 5, where we adopt the notation  $\Lambda_{i,j}^P = \Lambda_i^P \cup \Lambda_j^P$  and  $\Lambda_{i,j,k}^P = \Lambda_i^P \cup \Lambda_j^P \cup \Lambda_k^P$ . The graph  $G_P$  has totally 13 connected components, and among them there are four connected components  $C_1, C_2, C_3, C_4$  which have more than one vertex.

- To illustrate the assertion of Lemma 9, for example, let  $J = \Lambda_{4,5,6}^P$ , then we have  $gs(J) = \{4, 5, 6\}$ . One can verify that the subgraph  $\chi_J$  of  $G_P$  induced by the vertex subset  $\{\Lambda_4^P, \Lambda_5^P, \Lambda_6^P\}$  is indeed connected.

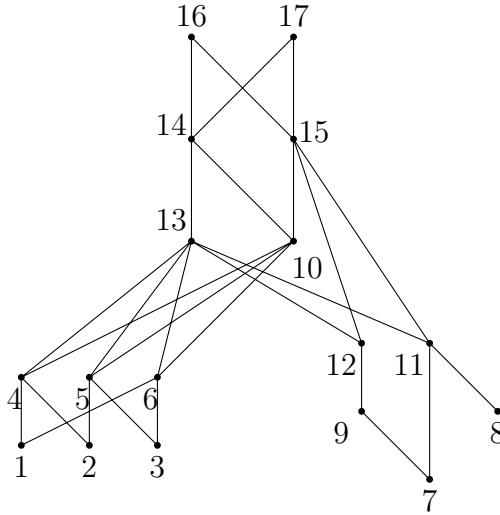


Figure 4: A naturally labeled poset  $P$ .

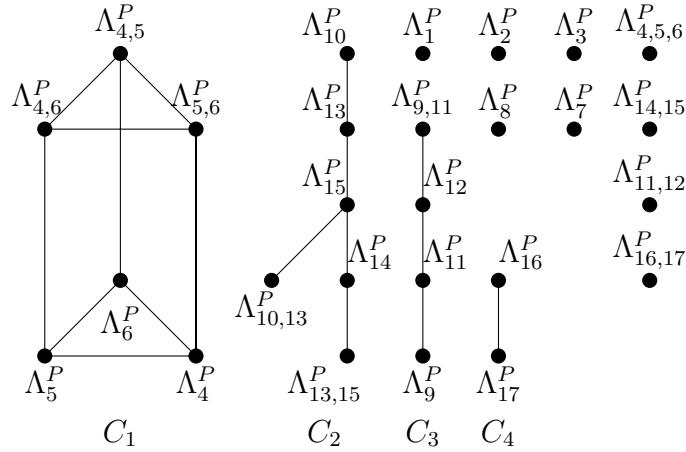


Figure 5: The graph  $G_P$  associated to the poset  $P$  in Figure 4.

- To illustrate the assertion of Lemma 10, for example, we let  $J = \Lambda_{10}^P$ , and let  $C$  be the connected component  $C_1$  of  $G_P$ , then  $\Lambda_5^P \subset J$ . In this case we see that  $J' \subset \Lambda_{10}^P$  for any  $J' \in V(C_1)$ .

Now we turn to study a special subgraph of  $G_P$ , which is induced by the principal order ideals of  $P$ . This graph also plays an important role in our future proofs. Recall that the set of principal order ideals of  $P$  consists of  $\Lambda_1^P, \Lambda_2^P, \dots, \Lambda_n^P$ . Let  $H_P$  be the subgraph of  $G_P$  induced by the vertex subset  $\{\Lambda_1^P, \Lambda_2^P, \dots, \Lambda_n^P\}$ . For example, for the poset  $P$  and the graph  $G_P$  as illustrated in Figures 4 and 5, the graph  $H_P$  is as shown in Figure 6. It follows from Lemma 9 that for a given connected order ideal  $J$  the induced subgraph  $\chi_J$

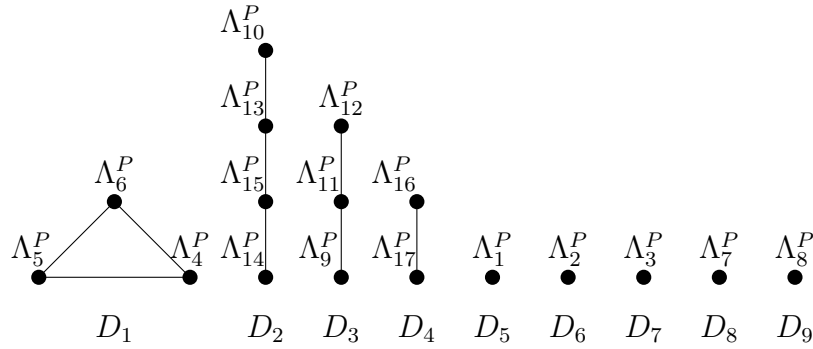


Figure 6: The subgraph  $H_P$  induced on  $G_P$  by principal order ideals.

must be a subgraph of certain connected component of  $H_P$ , where  $\chi_J$  is defined as before Lemma 9. The graph  $H_P$  admits the following interesting properties.

**Lemma 11.** *Suppose that  $H_P$  has connected components  $D_1, D_2, \dots, D_\ell$ . We have the following two assertions.*

- (1) *Let  $1 \leq r < s \leq \ell$ , and let  $J_a, J_b$  be two connected order ideals of  $P$ . If  $\chi_{J_a}$  is a subgraph of  $D_r$  while  $\chi_{J_b}$  is a subgraph of  $D_s$ , then  $J_a$  and  $J_b$  are not adjacent in  $G_P$ .*
- (2) *Given a connected order ideal  $J$ , suppose that  $\chi_J$  is a subgraph of the connected component  $D_r$  of  $H_P$ , and hence  $J \subseteq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$ . If  $J \neq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$ , then there exists some  $\Lambda_j^P \in V(D_r)$  such that  $J$  and  $\Lambda_j^P$  are adjacent in  $G_P$ .*

*Proof.* Let us first prove assertion (1). Suppose to the contrary that  $J_a$  and  $J_b$  are adjacent in the graph  $G_P$ . Then  $J_a \cap J_b \neq \emptyset$ . Since

$$J_a = \bigcup_{i \in gs(J_a)} \Lambda_i^P, \quad J_b = \bigcup_{j \in gs(J_b)} \Lambda_j^P,$$

there exist some  $i \in gs(J_a)$  and  $j \in gs(J_b)$  such that  $\Lambda_i^P \cap \Lambda_j^P \neq \emptyset$ . Notice that  $\Lambda_i^P$  is a vertex of the connected component  $D_r$  and  $\Lambda_j^P$  is a vertex of the connected component  $D_s$ , so  $\Lambda_i^P$  and  $\Lambda_j^P$  are not adjacent in the graph  $H_P$ . Since the graph  $H_P$  is a vertex induced subgraph of  $G_P$ , the order ideals  $\Lambda_i^P$  and  $\Lambda_j^P$  are also not adjacent in the graph  $G_P$ , hence they intersect trivially. Because  $\Lambda_i^P \cap \Lambda_j^P \neq \emptyset$ , we must have  $\Lambda_i^P \subset \Lambda_j^P$  or  $\Lambda_j^P \subset \Lambda_i^P$ . If  $\Lambda_i^P \subset \Lambda_j^P$ , by Lemmas 9 and 10 we obtain that for any  $k \in gs(J_a)$ , there is  $\Lambda_k^P \subset \Lambda_j^P$ . Then,

$$J_a = \bigcup_{k \in gs(J_a)} \Lambda_k^P \subset \Lambda_j^P \subseteq J_b,$$

which implies that  $J_a$  and  $J_b$  are not adjacent in the graph  $G_P$ . If  $\Lambda_j^P \subset \Lambda_i^P$ , we can use a similar argument to deduce that  $J_a$  and  $J_b$  are not adjacent in the graph  $G_P$ . In both cases, we are led to a contradiction.

We proceed to prove assertion (2). Recall that  $V(D_r)$  denotes the set of vertices of  $D_r$ . Assume that  $gs(J) = \{i_1, \dots, i_k\}$ . Since  $J \subseteq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$  but  $J \neq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$ , there exists some  $\Lambda_j^P \in V(D_r)$  such that  $\Lambda_j^P \not\subseteq J$ . Let

$$\begin{aligned} V_1 &= \{\Lambda_i^P \in V(D_r) \mid \Lambda_i^P \subseteq J\}, \\ V_2 &= \{\Lambda_j^P \in V(D_r) \mid \Lambda_j^P \not\subseteq J\}. \end{aligned}$$

Clearly, we have  $V_1 \cup V_2 = V(D_r)$  and  $V_2 \neq \emptyset$ . Since  $\chi_J$  is a subgraph of  $D_r$ , we see that  $V_1 \neq \emptyset$ . Because  $D_r$  is a connected component of  $H_P$ , there exist some  $\Lambda_i^P \in V_1$  and  $\Lambda_j^P \in V_2$  such that  $\Lambda_i^P$  and  $\Lambda_j^P$  are adjacent in the graph  $H_P$ . Since  $H_P$  is a vertex induced subgraph of  $G_P$ , the vertices  $\Lambda_i^P$  and  $\Lambda_j^P$  are also adjacent in  $G_P$ , which means that  $\Lambda_i^P$  and  $\Lambda_j^P$  intersect nontrivially, namely

$$\Lambda_i^P \cap \Lambda_j^P \neq \emptyset, \quad \Lambda_i^P \not\subseteq \Lambda_j^P, \quad \text{and} \quad \Lambda_j^P \not\subseteq \Lambda_i^P.$$

In view of that  $\Lambda_i^P \subseteq J$  and  $\Lambda_j^P \in V_2$ , we get  $J \neq \Lambda_j^P$  and

$$J \cap \Lambda_j^P \neq \emptyset, \quad J \not\subseteq \Lambda_j^P, \quad \text{and} \quad \Lambda_j^P \not\subseteq J.$$

Hence  $J$  is adjacent to  $\Lambda_j^P$ , as desired. □

With the above lemma, we can further obtain another property of  $G_P$ .

**Lemma 12.** *Let  $C_r$  be a connected component of  $G_P$  with vertex set  $V(C_r)$ . Let  $J$  be a connected order ideal with the graph  $\chi_J$  as defined as above. We have the following two assertions:*

- (1) *Let  $J_r^{max}$  denote the set  $\bigcup_{J' \in V(C_r)} J'$ . Then  $J_r^{max}$  is an isolated vertex of the graph  $G_P$ .*
- (2) *If  $\chi_J$  is a subgraph of  $C_r$ , and  $J \neq J_r^{max}$ , then  $J$  is a vertex of  $C_r$ .*

*Proof.* Let us first prove assertion (1). It is clearly true when  $|V(C_r)| = 1$ . Suppose  $|V(C_r)| \geq 2$ . We first prove that  $J_r^{max}$  is a connected order ideal. Let  $V$  be a set of connected order ideals and assume  $V$  satisfies the condition:

$$V \subseteq V(C_r) \text{ and } \bigcup_{J \in V} J \text{ is a connected order ideal.} \quad (*)$$

We claim that if  $V$  satisfies (\*) and is of the largest possible size, then  $V$  must be equal to  $V(C_r)$ . Otherwise, suppose  $V \subset V(C_r)$  but  $V \neq V(C_r)$ . Since  $C_r$  is a connected graph and  $|V(C_r)| \geq 2$ , there exist some  $J_a \in V$  and  $J_b \in (V(C_r) \setminus V)$  such that  $J_a$  and  $J_b$  are adjacent in  $G_P$ . Hence  $J_a \cap J_b \neq \emptyset$ , and then  $(\bigcup_{J \in V} J) \cap J_b \neq \emptyset$ . It follows that the set  $V' = V \cup \{J_b\}$  also satisfies the condition (\*), and  $|V'| = |V| + 1$ , contradicting the assumption that  $V$  is of the largest possible size.

We now prove that  $J_r^{max}$  is not adjacent to any other vertex of  $G_P$ . For a  $J \in \mathcal{J}_{conn}(P)$ , if  $J \in V(C_r)$ , then  $J \subset J_r^{max}$  and so  $J$  and  $J_r^{max}$  are not adjacent in  $G_P$ . If  $J \notin V(C_r)$ , namely,  $J$  is not adjacent to any vertex of  $C_r$ , we need to consider three cases:

- (i) There exists some  $J_a \in V(C)$  such that  $J_a \subset J$ . Then by Lemma 10 we obtain that  $J_b \subset J$  for any other  $J_b \in V(C_r)$ . Hence  $J_r^{max} \subset J$ , and it follows that  $J$  and  $J_r^{max}$  are not adjacent in  $G_P$ ;
- (ii) There exists some  $J_a \in V(C)$  such that  $J \subset J_a$ . Then  $J \subset J_r^{max}$ , and as a consequence,  $J$  and  $J_r^{max}$  are also not adjacent in  $G_P$ ;
- (iii)  $J \cap J_a = \emptyset$  for any  $J_a \in V(C_r)$ . Then  $J_r^{max} \cap J = \emptyset$  and, again,  $\tilde{J}$  and  $J$  are not adjacent in  $G_P$ .

Hence we conclude that  $J_r^{max}$  is an isolated vertex of the graph  $G_P$ .

To prove assertion (2), we first analyse some general properties of  $G_P$ . Suppose the graph  $H_P$  has  $\ell$  connected components  $D_1, D_2, \dots, D_\ell$ . Lemma 9 tells us that for any connected order ideal  $J'$ , the graph  $\chi_{J'}$  is connected, and that it must be a subgraph of  $D_k$  for some  $1 \leq k \leq \ell$ . For each  $1 \leq k \leq \ell$ , let

$$\mathcal{J}_{conn}^k(P) = \{J \in \mathcal{J}_{conn}(P) \mid \text{the graph } \chi_J \text{ is a subgraph of } D_k\}.$$

In particular, if  $J' = \Lambda_i^P \in V(D_k)$  is a principal order ideal, then the graph  $\chi_{J'}$  has only one vertex  $\Lambda_i^P$ , thus  $\chi_{J'}$  is of course a subgraph of  $D_k$ . It follows that  $V(D_k) \subseteq \mathcal{J}_{conn}^k(P)$  for each  $1 \leq k \leq \ell$ . It is clear that

$$\mathcal{J}_{conn}(P) = \mathcal{J}_{conn}^1(P) \uplus \mathcal{J}_{conn}^2(P) \uplus \dots \uplus \mathcal{J}_{conn}^\ell(P).$$

For each  $1 \leq k \leq \ell$ , let  $C_k$  be the connected component of  $G_P$  such that  $D_k$  is a subgraph of  $C_k$  (it turns out that for each  $D_k$ , there exists a unique  $C_k$  such that  $D_k$  is a subgraph of  $C_k$ ). We proceed to show that  $V(C_k) \subseteq \mathcal{J}_{conn}^k(P)$ . Note that if  $J_a \in \mathcal{J}_{conn}^s(P)$  and  $J_b \in \mathcal{J}_{conn}^t(P)$  for some  $s \neq t$ , the first assertion of Lemma 11 tells us that  $J_a$  and  $J_b$  are not adjacent in  $G_P$ . Thus, by the connectivity of  $C_k$  in  $G_P$ , all members of  $V(C_k)$  must belong to  $\mathcal{J}_{conn}^k(P)$  since we already have  $V(D_k) \subseteq \mathcal{J}_{conn}^k(P)$ . And then, we get that  $V(D_k) \subseteq V(C_k) \subseteq \mathcal{J}_{conn}^k(P)$ . That is to say, for any  $J' \in V(C_k)$ , the graph  $\chi_{J'}$  is a subgraph of  $D_k$ . Therefore,  $J' \subseteq \bigcup_{\Lambda_i^P \in V(D_k)} \Lambda_i^P$  for any  $J' \in V(C_k)$ . This leads to the following equality:

$$J_k^{max} = \bigcup_{J' \in V(C_k)} J' = \bigcup_{\Lambda_i^P \in V(D_k)} \Lambda_i^P. \tag{24}$$

For the given  $J$ , we assume that  $\chi_J$  is a subgraph of the connected component  $D_r$  of  $H_P$  for some  $1 \leq r \leq \ell$ , and then  $D_r$  is a subgraph of  $C_r$ . Thus in view of (24), when  $J \neq J_r^{max}$ , it follows that  $J \neq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$ . By the second assertion of Lemma 11, in the graph  $G_P$  we see that  $J$  is adjacent to some vertex of  $D_r$ , therefore,  $J$  is also a vertex of  $C_r$ .  $\square$

We are almost ready for the proof of Theorem 2. Note that the definition of  $\text{Des}(M)$  ( $M \in \mathcal{M}(G_P)$ ) is indirect, which uses the map  $\Psi$  from  $\mathcal{M}(G_P)$  to  $\mathcal{F}(P)$ . In order to make the proof of Theorem 2 more clear, we shall give another characterization of  $\text{Des}(M)$  which only uses the information of  $M$ . Before doing this, we shall introduce one more notation. Given  $J_a, J_b \in M$ , we say that  $J_a \prec_M J_b$  if  $J_a \subset J_b$  and there exists no  $J \in M$  such that  $J_a \subset J \subset J_b$ . Our new characterization of  $\text{Des}(M)$  is as follows.



**Lemma 13.** *Given  $M \in \mathcal{M}(G_P)$ , then  $i \in \text{Des}(M)$  if and only if there exists  $j < i$  such that  $J_a \prec_M J_b$ , where  $J_a, J_b \in M$  are connected order ideals uniquely determined by  $i, j$  respectively as in Lemma 7.*

*Proof.* By definition,  $i \in \text{Des}(M) = \text{Des}(F_M)$  if and only if the parent of  $i$ , say  $j$ , is greater than  $i$  with respect to the natural order on integers. Recall that if  $j$  is the parent of  $i$ , then  $i <_{F_M} j$  and there exists no  $k$  such that  $i <_{F_M} k <_{F_M} i$ . It follows from Lemma 7 that there exist two connected order ideals  $J_a, J_b$  in  $M$  satisfying  $J_a \setminus \mu(M, J_a) = \{i\}, J_b \setminus \mu(M, J_b) = \{j\}$ . By the construction of  $F_M$ , we have  $J_a \subset J_b$  but there exists no  $J \in M$  such that  $J_a \subset J \subset J_b$ , namely  $J_a \prec_M J_b$ .  $\square$

As shown above, the relation  $\prec_M$  plays an important role for the new characterization of  $\text{Des}(M)$ . To prove Theorem 2, we also need the following lemma, which is evident by definition. Recall that the set  $U_{max}(M, J)$  is defined by (12).

**Lemma 14.** *Given  $J_a, J_b \in M$ , if  $J_a \prec_M J_b$  then  $J_a \in U_{max}(M, J_b)$ .*

Now we are in the position to prove Theorem 2. From now on we shall assume that  $P$  is naturally labeled.

*Proof of Theorem 2.* There are two cases to consider.

(1). The connected component  $C_r$  has only one vertex, say  $J_r$ . Thus  $M_r$  can only be the unique one maximum independent set  $\{J_r\}$  of  $C_r$ . By Lemma 7, we have  $J_r \setminus \mu(M^1, J_r) = \{i\}$  for some  $i \in \{1, 2, \dots, n\}$ . In this case, we first prove that

$$\text{Des}(M_r, M^1) = \text{Des}(M_r, M^2) = \emptyset. \quad (25)$$

Otherwise, suppose that  $\text{Des}(M_r, M^1) = \{i\}$ . By the definition of  $\text{Des}(M_r, M^1)$ , we have  $i \in \text{Des}(M^1)$ . By Lemma 13, there exist  $j < i$  and  $J \in M^1$  such that  $J \setminus \mu(M^1, J) = \{j\}$  and  $J_r \prec_{M^1} J$ .

We proceed to show that it is impossible to have such a pair  $(i, j)$ . Let us consider the order relation between  $i$  and  $j$  in the poset  $P$ . It cannot be  $j <_P i$ , since  $i \in J_r \subset J$  and Lemma 6 tells us that  $j$  is a maximal element of  $J$ . Then it might be  $i <_P j$ , or  $i$  and  $j$  are incomparable in  $P$ . Since  $P$  is naturally labeled and  $j < i$ , it can not be  $i <_P j$ . Suppose that  $i$  and  $j$  are incomparable in  $P$ . Since  $J_r \setminus \mu(M^1, J_r) = \{i\}$ , it follows from Lemma 6 that  $i$  is a maximal element of  $J_r$ . We proceed to prove that  $i$  is also a maximal elements of  $J$ . To see this, it is enough to show that there exists no  $k \in J$  satisfying  $i <_P k$ . Note that

$$J = \{j\} \cup \mu(M^1, J) = \{j\} \cup \left( \bigcup_{J' \in U(M^1, J)} J' \right) = \{j\} \cup \left( \bigcup_{J' \in U_{max}(M^1, J)} J' \right).$$

By Lemma 14, the relation  $J_r \prec_{M^1} J$  implies that  $J_r \in U_{max}(M^1, J)$ . Then there are three cases to consider:

- (i) If  $k = j$ , then  $i$  and  $k$  are incomparable in  $P$ ;

- (ii) If  $k \in J_r$ , in this case we have  $k \leq_P i$ , or  $i$  and  $k$  are incomparable in  $P$ , because  $i$  is a maximal element of  $J_r$ ;
- (iii) If  $k \in J'$  for some  $J' \in U_{max}(M^1, J)$  but  $J' \neq J_r$ , we obtain that  $i$  and  $k$  are incomparable in  $P$ , since by Lemma 4 we have  $J' \cap J_r = \emptyset$ , which implies that for any  $u \in J_r, v \in J', u$  and  $v$  are incomparable in  $P$ .

Hence there exists no  $k \in J$  such that  $i <_P k$ , i.e.,  $i$  is a maximal element of  $J$ . It follows that  $\{i, j\} \subseteq gs(J)$  and then the graphs  $\chi_{J_r}$  and  $\chi_J$  have a common vertex  $\Lambda_i^P$ . Then by Lemma 9, the graphs  $\chi_{J_r}$  and  $\chi_J$  belong to the same connected component  $C_s$  of  $G_P$ . Hence  $C_s$  has at least two vertices  $\Lambda_i^P$  and  $\Lambda_j^P$ . By Lemma 12 and the hypothesis that  $J_r$  is an isolated vertex of  $G_P$ , we obtain  $J_r = \bigcup_{J' \in V(C_s)} J'$  and  $J \subseteq \bigcup_{J' \in V(C_s)} J'$ . This contradicts with the assumption that  $J_r \prec_{M^1} J$ . Hence  $i$  and  $j$  cannot be incomparable in  $P$ , a contradiction.

Since such a pair  $(i, j)$  can not exist, it follows that  $\text{Des}(M_r, M^1) = \emptyset$ . By using a similar argument, one can also prove that  $\text{Des}(M_r, M^2) = \emptyset$ . Moreover, by the definition of  $\overline{\text{Des}}(M_r, M)$ , it is clear that

$$\overline{\text{Des}}(M_r, M^1) = \overline{\text{Des}}(M_r, M^2) = \emptyset.$$

(2).  $C_r$  has at least two vertices. In this case,  $M_r \subset V(C_r)$ . By Lemma 12, we see that  $J_r^{max} = \bigcup_{J' \in V(C_r)} J'$  is an isolated vertex of  $G_P$ . Hence  $J_r^{max} \in M$  holds for any maximum independent set of  $G_P$ , and in particular  $J_r^{max} \in M^1$  as well as  $J_r^{max} \in M^2$ .

We first prove that for any  $J \in M_r$  or  $J = J_r^{max}$ ,

$$J \setminus \mu(M^1, J) = J \setminus \mu(M^2, J). \tag{26}$$

To see this, we partition the set  $U(M^2, J)$  into two subsets  $B_1$  and  $B_2$ , where

$$\begin{aligned} B_1 &= \{J_1 \in U(M^2, J) \mid J_1 \in V(C_r)\}, \\ B_2 &= \{J_2 \in U(M^2, J) \mid J_2 \notin V(C_r)\}. \end{aligned}$$

Assume  $J \setminus \mu(M^1, J) = \{j\}$ . We claim that  $j \notin J_2$  for any  $J_2 \in B_2$ . Otherwise, suppose to the contrary that there exists some  $J_2 \in B_2$  such that  $j \in J_2$ . It follows from Lemma 6 that  $j \in gs(J)$ . On the other hand, since  $J_2 \subset J$ , we obtain that  $j \in gs(J_2)$ . Hence the graph  $\chi_J$  and  $\chi_{J_2}$  have a common vertex  $\Lambda_j^P$ . Then by Lemma 9 the graphs  $\chi_J$  and  $\chi_{J_2}$  belong to the same connected component of  $G_P$ . We proceed to show that  $\chi_{J_2}$  is a subgraph of  $C_r$ . To see this, there are two cases to consider.

- (i) Suppose that  $J \in M_r \subset V(C_r)$  (then  $J \neq J_r^{max}$ ), namely,  $J$  is a vertex of the connected component  $C_r$ . It follows from the second assertion of Lemma 12 that  $\chi_J$  and  $J$  are contained in the same connected component  $C_r$  of  $G_P$ . Hence both  $\chi_J$  and  $\chi_{J_2}$  are subgraphs of  $C_r$ .

(ii) Suppose that  $J = J_r^{max} = \bigcup_{J' \in V(C_r)} J'$ . Let  $i \in gs(J)$  be a maximal element of  $J$ , then there exists some  $J' \in V(C_r)$  such that  $i \in J'$ . It follows that  $i$  is also a maximal element of  $J'$ , namely,  $i \in gs(J')$ . Hence the graphs  $\chi_J$  and  $\chi_{J'}$  have at least one common vertex  $\Lambda_i^P$ , and then  $\chi_J$  and  $\chi_{J'}$  belong to the same connected component of  $G_P$ . The second assertion of Lemma 12 tells us that for any  $J' \in V(C_r)$ ,  $\chi_{J'}$  and  $J'$  are contained in the same connected component  $C_r$  of  $G_P$ . Hence  $\chi_J, \chi_{J'}$  and  $\chi_{J_2}$  are all subgraphs of  $C_r$ .

On the other hand, because  $J_2 \subset J$ , we have  $J_2 \neq J_r^{max}$ . Then by the second assertion of Lemma 12 we get  $J_2 \in V(C_r)$ , leading to a contradiction. Hence the claim, that  $j \notin J_2$  for any  $J_2 \in B_2$ , is true.

Recall that  $M^1 \cap V(C_r) = M^2 \cap V(C_r) = M_r$ . It is routine to verify that

$$U(M^1, J) \cap M_r = U(M^2, J) \cap M_r = B_1,$$

Combining (13) and the above identity, we get that

$$j \in J \setminus \mu(M^1, J) \subseteq J \setminus \bigcup_{J_1 \in B_1} J_1.$$

As we have shown that  $j \notin J_2$  for any  $J_2 \in B_2$ , so again by (13) there holds

$$j \in J \setminus \bigcup_{J' \in (B_1 \cup B_2)} J' = J \setminus \bigcup_{J' \in U(M^2, J)} J' = J \setminus \mu(M^2, J).$$

Thus, by Lemma 6, the set  $J \setminus \mu(M^2, J)$  contains exactly one element, which can only be  $j$ . Therefore, we have

$$\{j\} = J \setminus \mu(M^2, J) = J \setminus \mu(M^1, J).$$

We proceed to show that  $\text{Des}(M_r, M^1) \subseteq \text{Des}(M_r, M^2)$ . Let  $i \in \text{Des}(M_r, M^1)$ , and by the definition of  $\text{Des}(M_r, M^1)$  and Lemma 6 there exists  $J_a \in M_r$  such that  $J_a \setminus \mu(M^1, J_a) = \{i\}$ . By Lemma 13, there exist  $j < i$  and  $J_b \in M^1$  such that  $J_b \setminus \mu(M^1, J_b) = \{j\}$  and  $J_a \prec_{M^1} J_b$ . We claim that  $J_b \in V(C_r)$  or  $J_b = J_r^{max}$ . Suppose otherwise that  $J_b$  is not a vertex of  $C_r$  and  $J_b \neq J_r^{max}$ . Since  $J_a \in V(C_r)$  and  $J_a \subset J_b$ , it follows from Lemma 10 that  $J' \subset J_b$  for any  $J' \in V(C_r)$ . Hence  $J_r^{max} \subset J_b$ . Thus we obtain  $J_a \subset J_r^{max} \subset J_b$ . Recall that  $J_r^{max} \in M^1$ , the relation  $J_a \subset J_r^{max} \subset J_b$  contradicts the assumption that  $J_a \prec_{M^1} J_b$ . Recall also that we have shown  $J_r^{max} \in M^2$ . If  $J_b = J_r^{max}$  then  $J_b \in M^2$ . If  $J_b \in V(C_r)$ , then  $J_b \in M_r = M^2 \cap V(C_r)$ , and hence also  $J_b \in M^2$ . We further show that  $J_a \prec_{M^2} J_b$ . Otherwise, suppose there exists some  $J_c \in M^2$  such that  $J_a \subset J_c \subset J_b$ . By the hypothesis that  $J_a \prec_{M^1} J_b$  and  $M^1 \cap V(C_r) = M^2 \cap V(C_r) = M_r$ , it follows that  $J_c \notin M_r \subset V(C_r)$ . Then by Lemma 10, for any  $J' \in V(C_r)$ , there is  $J' \subset J_c$ . Hence  $J_b \subseteq \bigcup_{J' \in V(C_r)} J' \subset J_c$ , leading to a contradiction. Thus, for any  $i \in \text{Des}(M_r, M^1)$ , by (26) there exist  $J_a, J_b \in M^2$  such that  $J_a \setminus \mu(M^2, J_a) = \{i\}$ ,  $J_b \setminus \mu(M^2, J_b) = \{j\}$ ,  $J_a \prec_{M^2} J_b$  and  $i > j$ . This means  $i \in \text{Des}(M_r, M^2)$  for any  $i \in \text{Des}(M_r, M^1)$ . Hence  $\text{Des}(M_r, M^1) \subseteq \text{Des}(M_r, M^2)$ .

It can be proved in a similar way that  $\text{Des}(M_r, M^2) \subseteq \text{Des}(M_r, M^1)$ . So we get  $\text{Des}(M_r, M^1) = \text{Des}(M_r, M^2)$ . Combining this and (26), we further obtain  $\overline{\text{Des}}(M_r, M^1) = \overline{\text{Des}}(M_r, M^2)$ , as desired.  $\square$

We proceed to prove Theorem 3.

*Proof of Theorem 3.* Given a maximum independent set  $M$  of  $G_P$ , let

$$\overline{\text{Des}}(M) = \{J \in M \mid J \setminus \mu(M, J) = \{i\} \text{ for some } i \in \text{Des}(M)\}.$$

Recall that  $\mathcal{M}(C_r)$  is the set of maximum independent sets of  $C_r$  for each  $1 \leq r \leq h$ , respectively. It is clear that  $M$  admits the following natural decomposition:

$$M = M_1 \uplus M_2 \uplus \cdots \uplus M_h, \text{ where } M_r \in \mathcal{M}(C_r).$$

It follows from Theorem 2 that both  $\text{Des}(M_r)$  and  $\overline{\text{Des}}(M_r)$  are well-defined, and hence

$$\text{Des}(M) = \text{Des}(M_1) \uplus \text{Des}(M_2) \uplus \cdots \uplus \text{Des}(M_h), \quad (27)$$

$$\overline{\text{Des}}(M) = \overline{\text{Des}}(M_1) \uplus \overline{\text{Des}}(M_2) \uplus \cdots \uplus \overline{\text{Des}}(M_h). \quad (28)$$

Thus, by (6), Theorem 1 and Lemma 7, we get that

$$F_P(\mathbf{x}) = \sum_{M \in \mathcal{M}(G_P)} \frac{\prod_{J \in \overline{\text{Des}}(M)} \prod_{k \in J} x_k}{\prod_{J \in M} (1 - \prod_{\ell \in J} x_\ell)}.$$

By (28), we then have

$$\begin{aligned} F_P(\mathbf{x}) &= \sum_{M_1 \in \mathcal{M}(C_1)} \sum_{M_2 \in \mathcal{M}(C_2)} \cdots \sum_{M_h \in \mathcal{M}(C_h)} \frac{\prod_{r=1}^h \prod_{J \in \overline{\text{Des}}(M_r)} \prod_{k \in J} x_k}{\prod_{r=1}^h \prod_{J \in M_r} (1 - \prod_{\ell \in J} x_\ell)} \\ &= \prod_{r=1}^h \sum_{M_r \in \mathcal{M}(C_r)} \frac{\prod_{J \in \overline{\text{Des}}(M_r)} \prod_{k \in J} x_k}{\prod_{J \in M_r} (1 - \prod_{\ell \in J} x_\ell)}. \end{aligned} \quad \square$$

We would like to point out that Theorem 3 enables us to give an alternative proof to Féray and Reiner's formula (4). To this end, let  $P$  be a naturally labeled forest with duplications as defined by Féray and Reiner [4], namely, for any connected order ideal  $J_a$  of  $P$ , there exists at most one other connected order ideal  $J_b$  such that  $J_a$  and  $J_b$  intersect nontrivially. Assume that  $G_P$  has  $h$  connected components  $C_1, C_2, \dots, C_h$ . Then each  $C_r$  has at most two vertices, and hence each connected component of  $H_P$  has also at most two vertices.

We claim that when a connected component  $C$  of  $G_P$  has two vertices, say  $J_a$  and  $J_b$ , then both  $J_a$  and  $J_b$  are principal order ideals of  $P$ . Otherwise, suppose that  $J_a$  is not a principal order ideal of  $P$ . Then the graph  $\chi_{J_a}$  has more than one vertices. Recall that  $\chi_{J_a}$  is a subgraph of  $H_P$ . By Lemma 9 and the fact that each connected component of the graph  $H_P$  has at most two vertices, the graph  $\chi_{J_a}$  is a connected component of  $H_P$ .

It then follows from (24) and the first assertion of Lemma 12 that  $J_a$  is an isolated vertex of  $G_P$ , a contradiction. Similarly,  $J_b$  is also a principal order ideal of  $P$ .

Therefore, we may assume that for  $1 \leq r \leq d$  the component  $C_r$  has two vertices (both of them are principal order ideals of  $P$ ), say  $\Lambda_{i_r}^P$  and  $\Lambda_{j_r}^P$ , and for  $d < r \leq h$  the component  $C_r$  has only one vertex. Thus, for  $1 \leq r \leq d$ , there are two choices for  $M_r$ , namely,  $M_r = \{\Lambda_{i_r}^P\}$  or  $M_r = \{\Lambda_{j_r}^P\}$ . We assume that  $i_r > j_r$ . Then

$$\overline{\text{Des}}(\{\Lambda_{i_r}^P\}) = \Lambda_{i_r}^P, \quad \overline{\text{Des}}(\{\Lambda_{j_r}^P\}) = \emptyset.$$

For  $d < r \leq h$ , let  $J_r$  be the only vertex of  $C_r$ , and then  $\overline{\text{Des}}(\{J_r\}) = \emptyset$ . By Theorem 3, we obtain that

$$\begin{aligned} F_P(\mathbf{x}) &= \prod_{1 \leq r \leq d} \left[ \frac{\mathbf{x}^{\Lambda_{i_r}^P}}{(1 - \mathbf{x}^{\Lambda_{i_r}^P})} + \frac{1}{(1 - \mathbf{x}^{\Lambda_{j_r}^P})} \right] \prod_{d < r \leq h} \frac{1}{(1 - \mathbf{x}^{J_r})} \\ &= \prod_{1 \leq r \leq d} \left[ \frac{1 - \mathbf{x}^{\Lambda_{i_r}^P} \mathbf{x}^{\Lambda_{j_r}^P}}{(1 - \mathbf{x}^{\Lambda_{i_r}^P})(1 - \mathbf{x}^{\Lambda_{j_r}^P})} \right] \prod_{d < r \leq h} \frac{1}{(1 - \mathbf{x}^{J_r})}, \end{aligned}$$

where  $\mathbf{x}^A = \prod_{i \in A} x_i$  for a subset  $A \subseteq \{1, 2, \dots, n\}$ . It is straightforward to verify that the above formula is equivalent to (4).

## 4 Counting linear extensions

In this section, we take an example to show that formula (11) can be used to derive the generating function of major index of linear extensions of  $P$ , as well as to count the number  $|\mathcal{L}(P)|$  of linear extensions of  $P$ .

The generating function  $F_P(q)$  of major index of linear extensions of  $P$  is denoted by  $F_P(q) = \sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)}$ , where  $\text{maj}(w) = \sum_{i \in \text{Des}(w)} i$  is called the major index of  $w$ . By letting  $x_1 = \dots = x_n = q$  respectively in (1) and (11), we are led to the following identity

$$F_P(q) = [n]!_q \prod_{r=1}^h \sum_{M_r \in \mathcal{M}(C_r)} \frac{q^{\sum_{J \in \overline{\text{Des}}(M_r)} |J|}}{\prod_{J \in M_r} [J]_q}, \quad (29)$$

where  $[i]_q = 1 - q^i$  for any  $i$  and  $[m]!_q = \prod_{i=1}^m [i]_q$ .

Moreover, when  $q$  tends to 1 on both sides of (29), we arrive at the following formula for the number of linear extensions of  $P$ :

$$|\mathcal{L}(P)| = n! \prod_{r=1}^h \sum_{M_r \in \mathcal{M}(C_r)} \frac{1}{\prod_{J \in M_r} |J|}. \quad (30)$$

Note that the number of linear extensions of  $P$  is independent of the labelling of  $P$ . Thus formula (30) is also valid in the cases when  $P$  is not naturally labeled.

We would like to mention that calculating the number of linear extensions for general posets has been proved to be a  $\sharp P$ -hard problem by Brightwell and Winkler [3]. However, in the case when  $P$  is a poset such that each connected component  $C_r$  of  $G_P$  has small size of vertex set, we shall illustrate that formula (30) provides an efficient way to count the number of linear extensions of  $P$ . For example, take the naturally labeled poset  $P$  in Figure 4. From the graph of  $G_P$  as illustrated in Figure 5, we obtain that

1. For the connected component  $C_1$ , there are 6 choices for  $M_1$ :

$M_1$	$\{\Lambda_4^P, \Lambda_{4,5}^P\}$	$\{\Lambda_4^P, \Lambda_{4,6}^P\}$	$\{\Lambda_5^P, \Lambda_{4,5}^P\}$	$\{\Lambda_5^P, \Lambda_{5,6}^P\}$
$\text{Des}(M_1)$	$\emptyset$	$\{6\}$	$\{5\}$	$\{6\}$
$\overline{\text{Des}}(M_1)$	$\emptyset$	$\{\Lambda_{4,6}^P\}$	$\{\Lambda_5^P\}$	$\{\Lambda_{5,6}^P\}$

$M_1$	$\{\Lambda_6^P, \Lambda_{4,6}^P\}$	$\{\Lambda_6^P, \Lambda_{5,6}^P\}$
$\text{Des}(M_1)$	$\{6\}$	$\{5,6\}$
$\overline{\text{Des}}(M_1)$	$\{\Lambda_6^P\}$	$\{\Lambda_6^P, \Lambda_{5,6}^P\}$

2. For the connected component  $C_2$ , there are 5 choices for  $M_2$ :

$M_2$	$\{\Lambda_{10}^P, \Lambda_{15}^P, \Lambda_{13,15}^P\}$	$\{\Lambda_{10}^P, \Lambda_{10,13}^P, \Lambda_{14}^P\}$	$\{\Lambda_{10}^P, \Lambda_{10,13}^P, \Lambda_{13,15}^P\}$
$\text{Des}(M_2)$	$\{15\}$	$\emptyset$	$\{15\}$
$\overline{\text{Des}}(M_2)$	$\{\Lambda_{15}^P\}$	$\emptyset$	$\{\Lambda_{13,15}^P\}$

$M_2$	$\{\Lambda_{13}^P, \Lambda_{10,13}^P, \Lambda_{14}^P\}$	$\{\Lambda_{13}^P, \Lambda_{10,13}^P, \Lambda_{13,15}^P\}$
$\text{Des}(M_2)$	$\{13\}$	$\{13, 15\}$
$\overline{\text{Des}}(M_2)$	$\{\Lambda_{13}^P\}$	$\{\Lambda_{13}^P, \Lambda_{13,15}^P\}$

3. For the connected component  $C_3$ , there are 3 choices for  $M_3$ :

$M_3$	$\{\Lambda_{11}^P, \Lambda_{11,9}^P\}$	$\{\Lambda_9^P, \Lambda_{11,9}^P\}$	$\{\Lambda_9^P, \Lambda_{12}^P\}$
$\text{Des}(M_3)$	$\{11\}$	$\emptyset$	$\{12\}$
$\overline{\text{Des}}(M_3)$	$\{\Lambda_{11}^P\}$	$\emptyset$	$\{\Lambda_{12}^P\}$

4. For the connected component  $C_4$ , there are 2 choices for  $M_4$ :

$M_4$	$\{\Lambda_{16}^P\}$	$\{\Lambda_{17}^P\}$
$\text{Des}(M_4)$	$\emptyset$	$\{17\}$
$\overline{\text{Des}}(M_4)$	$\emptyset$	$\{\Lambda_{17}^P\}$

5. For connected components which have only one vertex, each of them has only one choice for each  $M_r$ , and  $\text{Des}(M_r) = \emptyset$  as well as  $\overline{\text{Des}}(M_r) = \emptyset$ .

Therefore, invoking formula (29), we see that  $F_P(q) = \sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)}$  equals

$$\begin{aligned}
 & [17]_q! \left[ \frac{1}{[6]_q} \left( \frac{1 + 2q^3 + 2q^5 + q^8}{[3]_q [5]_q} \right) \right] \left[ \frac{1}{[15]_q} \left( \frac{q^{13} + 1 + q^{14}}{[7]_q [13]_q [14]_q} + \frac{q^{12} + q^{26}}{[12]_q [13]_q [14]_q} \right) \right] \\
 & \quad \times \left[ \frac{1}{[5]_q} \left( \frac{q^3}{[3]_q [4]_q} + \frac{1}{[2]_q [4]_q} + \frac{q^3}{[2]_q [3]_q} \right) \right] \left[ \frac{1}{[17]_q} \frac{(1 + q^{16})}{[16]_q} \right] \times 1^5.
 \end{aligned}$$

Letting  $q \rightarrow 1$  in the above formula, we arrive at

$$\begin{aligned} |\mathcal{L}(P)| &= 17! \times \left( \frac{1}{6} \times \frac{6}{3 \times 5} \right) \times \left[ \frac{1}{15} \times \left( \frac{3}{7 \times 13 \times 14} + \frac{2}{13 \times 12 \times 14} \right) \right] \\ &\quad \times \left[ \frac{1}{5} \times \left( \frac{1}{3 \times 4} + \frac{1}{3 \times 2} + \frac{1}{4 \times 2} \right) \right] \times \left( \frac{1}{17} \times \frac{2}{16} \right) \times 1^5 \\ &= 2851200. \end{aligned}$$

This coincides with the result by listing all linear extensions by using Sage [10].

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