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Rainbow connection numbers of Cayley graphs

Yingbin Ma[1,2] · **Zaiping Lu[2]**

Abstract An edge colored graph is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number, rc-number for short, of a graph Γ, is the smallest number of colors that are needed in order to make Γ rainbow connected. In this paper, we give a method to bound the rc-numbers of graphs with certain structural properties. Using this method, we investigate the rc-numbers of Cayley graphs, especially, those defined on abelian groups and on dihedral groups.

Keywords Edge-coloring · Rainbow connection number · Cayley graph · Dihedral group · Interconnection networks

Mathematics Subject Classification 05C15 · 05C40

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty (2008) for those not described here.

✉ Yingbin Ma
  mayingbincw@htu.cn

  Zaiping Lu
  lu@nankai.edu.cn

[1] College of Mathematics and Information Science, Henan Normal University, XinXiang 453007, People's Republic of China

[2] Center for Combinatorics LPMC-TJKLC, Nankai University, Tianjin 300071, People's Republic of China

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For a graph Γ, we denote by $V\Gamma$ and $E\Gamma$ the vertex set and edge set of Γ, respectively. An *edge-coloring* of a graph Γ is a mapping from $E\Gamma$ to some finite set of colors. A path in an edge colored graph is said to be a *rainbow path* if no two edges on this path share the same color. An edge colored graph Γ is *rainbow connected* if each pair of distinct vertices of Γ are joined by some rainbow path, while the coloring is called a *rainbow coloring*. The *rainbow connection number* of a connected graph, *rc-number* for short, is the smallest number of colors that are needed in order to make the graph rainbow connected. For a connected graph Γ, we denote by $rc(\Gamma)$ its *rc*-number. Then, for a connected graph Γ with diameter $diam(\Gamma)$, it follows from the definition that

$$diam(\Gamma) \leq rc(\Gamma) \leq |E\Gamma|.$$

Moreover, $rc(\Gamma) \leq rc(\Sigma)$ for each connected spanning subgraph Σ of a graph Γ.

The concept of *rc*-number was introduced in Chartrand et al. (2008), where the *rc*-numbers of several graph classes were determined. Since then the study of *rc*-numbers has received considerable attention in the literature, see Li and Sun (2012) for a survey on this topic. It was shown in Chakraborty et al. (2011) that computing the *rc*-number of an arbitrary graph is an NP-Hard problem. Subsequently, there have been various investigations towards finding good upper (or lower) bounds for *rc*-numbers in terms of graph parameters such as connectivity, minimum degree, radius etc., see Basavaraju et al. (2014), Caro et al. (2008), Chandran et al. (2012), Krivelevich and Yuster (2009), Li et al. (2012), and Schiermeyer (2009) for example. In particular, it was shown in Li et al. (2012) that a 2-connected graph of order n has *rc*-number no more than $\lceil \frac{n}{2} \rceil$. In this paper, we focus our attention on a special class of 2-connected graphs.

Let G be a finite group with identity element 1, and let S be a subset of G such that $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$. The *Cayley graph* $\mathrm{Cay}(G, S)$ (on G with respect to S) is defined on G such that two 'vertices' g and h are adjacent if and only if $g^{-1}h \in S$. Then $\mathrm{Cay}(G, S)$ is a well-defined simple regular graph of valency $|S|$. It is well-known that $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$, that is, S is a generating set of the underlying group G. A subset X of G is a *minimal generating set* if G is generated by X but not by any proper subset of X.

Cayley graphs have been an active topic in algebraic graph theory for a long time. In fact, interconnection networks are often modeled by highly symmetric Cayley graphs (Akers and Krishnamurthy 1989). The rainbow connection number of a graph can be applied to measure the safety of a network. Thus the object of the rainbow connection numbers of Cayley graphs should be meaningful. Li et al. (2011) established an upper bound for the *rc*-numbers of Cayley graphs on abelian groups by using minimal generating sets. For an element $x \in G$, denote by $|x|$ the order of x in G.

Theorem 1.1 (Li et al. 2011) *Let G be a finite abelian group, and let S be a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then $rc(\Gamma) \leq \sum_{x \in X} \lceil \frac{|x|}{2} \rceil$, where X is an arbitrary minimal generating set of G contained in S.*

This motivates us to consider the *rc*-numbers of Cayley graphs, especially, those defined on non-abelian groups. In Sect. 2 we establish a method to bound the *rc*-numbers of (Cayley) graphs satisfying certain structural properties, which leads to a

new proof and an improvement of Theorem 1.1. In Sect. 3, applying the method given in Sect. 2, we investigate the rc-numbers of connected Cayley graphs on dihedral groups.

2 rc-numbers of Cayley graphs on abelian groups

Let Γ be a graph. For V_1, $V_2 \subseteq V\Gamma$, we denote by $\Gamma[V_1, V_2]$ the subgraph on $V_1 \cup V_2$ with edge set $\{\{v_1, v_2\} \in E\Gamma \mid v_1 \in V_1, v_2 \in V_2\}$. For a partition $\mathcal{B} = \{V_0, V_1, \ldots, V_{m-1}\}$ of $V\Gamma$, define a graph $\Gamma_\mathcal{B}$ on $\mathcal{B}$ such that distinct 'vertices' V_i and V_j are adjacent if and only if some $u \in V_i$ is adjacent to some $v \in V_j$ in Γ. The graph $\Gamma_\mathcal{B}$ is called a *quotient graph* of Γ. We first prove a technical lemma.

Lemma 2.1 *Let Γ be a connected graph. Assume that $V\Gamma$ has a partition $\mathcal{B} = \{V_0, V_1, \ldots, V_{m-1}\}$ such that, for each i, the subgraph $\Gamma[V_i, V_i]$ is connected.*

(i) *Suppose that, for $0 \leq i < m$, $\Gamma[V_i, V_{i+1}]$ is not empty, and every $u \in V_i$ is adjacent to some $v \in V_{i-1}$ or some $w \in V_{i+1}$ in Γ, reading the subscripts modulo m. Then*

$$rc(\Gamma) \leq (\max\{rc(\Gamma[V_i, V_i]) \mid 0 \leq i < m\} + 1)\left\lceil \frac{m}{2} \right\rceil.$$

(ii) *Suppose that, for $0 \leq i, j < m$, every subgraph $\Gamma[V_i, V_j]$ has no isolated vertices provided that it has at least one edge. Then*

$$rc(\Gamma) \leq \max\{rc(\Gamma[V_i, V_i]) \mid 0 \leq i < m\} + rc(\Gamma_\mathcal{B}).$$

Proof Let $\Gamma_i = \Gamma[V_i, V_i]$ and $c = \max\{rc(\Gamma_i) \mid 0 \leq i < m\}$.

(i) Now we are ready to show that Γ has a connected spanning subgraph which has rc-number no more than $(c + 1)\lceil \frac{m}{2} \rceil$. Consider the spanning subgraph Σ of Γ with edge set

$$E\Sigma = (\cup_{i=0}^{m-1} E\Gamma_i) \cup (\cup_{i=0}^{m-1} E\Gamma[V_i, V_{i+1}]),$$

where $V_m = V_0$. (Note that $\Sigma_\mathcal{B}$ is a cycle of length m.) Let C_0, C_1, $\ldots$, C_{m-1} be c-sets of colors such that $C_i \cap C_j = \emptyset$ if $0 \leq i < j < \lceil \frac{m}{2} \rceil$, and $C_i = C_j$ if $i \equiv j \pmod{\lceil \frac{m}{2} \rceil}$. For each graph Γ_i, since $rc(\Gamma_i) \leq c$, we choose a rainbow coloring $\theta_i : E\Gamma_i \to C_i$. Choose $\lceil \frac{m}{2} \rceil$ colors c_1, c_2, $\ldots$, $c_{\lceil \frac{m}{2} \rceil}$ which are not used above. We define an edge-coloring θ of Σ as follows:

$$\theta(e) = \begin{cases} \theta_i(e) & \text{if } e \in E\Gamma_i \text{ for } 0 \leq i < m; \\ c_i & \text{if } e \in E\Gamma[V_{i-1}, V_i] \text{ for } 1 \leq i \leq \lceil \frac{m}{2} \rceil; \\ c_{j-\lceil \frac{m}{2} \rceil} & \text{if } e \in E\Gamma[V_{j-1}, V_j] \text{ for } \lceil \frac{m}{2} \rceil < j \leq m. \end{cases}$$

It is straightforwardly checked that Γ is rainbow connected with the edge-coloring θ. Then the lemma follows from enumerating the number of colors used for θ.

(ii) Let C be a set of c colors and D be a set of $rc(\Gamma_{\mathcal{B}})$ colors with $C \cap D = \emptyset$. For each graph Γ_i, choose a rainbow coloring $\theta_i : E\Gamma_i \to C$. For $\Gamma_{\mathcal{B}}$, we choose a rainbow coloring $\bar{\theta} : E\Gamma_{\mathcal{B}} \to D$. We define an edge-coloring θ of Γ as follows:

$$\theta(e) = \begin{cases} \theta_i(e) & \text{if } e \in E\Gamma_i \text{ for } 0 \le i < m; \\ \bar{\theta}(\{V_i, V_j\}) & \text{if } \{V_i, V_j\} \in E\Gamma_{\mathcal{B}} \text{ and } e \in E\Gamma[V_i, V_j]. \end{cases}$$

Then Γ is rainbow connected with the edge-coloring θ. Thus

$$rc(\Gamma) \le |C \cup D| = c + rc(\Gamma_{\mathcal{B}}) = \max\{rc(\Gamma_i) \mid 0 \le i < m\} + rc(\Gamma_{\mathcal{B}}).$$

$\square$

Let G be a group and N a normal subgroup of G. Then all (left) cosets of N in G form a group under the product

$$(gN)(hN) = ghN,$$

which is denoted by G/N and called *the quotient group* of G with respect to N.

Theorem 2.2 *Let G be a finite group and S a generating set of G such that $1 \notin S = S^{-1}$. Suppose that $X \subseteq S$ such that $N := \langle S \setminus (X \cup X^{-1}) \rangle \ne G$. Set $\Gamma = \mathrm{Cay}(G, S)$, $Y = S \setminus (X \cup X^{-1})$ and $\Sigma = \mathrm{Cay}(N, Y)$. Suppose that N is normal in G. Then $rc(\Gamma) \le rc(\Sigma) + rc(\mathrm{Cay}(\bar{G}, \bar{X}))$, where $\bar{G} = G/N$ and $\bar{X} = \{xN \mid x \in S \setminus N\}$.*

Proof Since N is normal in G, we have $G = \langle X, Y \rangle \le \langle X, N \rangle = \langle X \rangle N$, and so $G = \langle X \rangle N$. Let m be the index of N in G. Then $m = \frac{|G|}{|N|}$. Let $g_0 N = N, g_1 N, \ldots g_i N, \ldots, g_{m-1} N$ be all distinct left cosets of N in G. Set $V_i = g_i N$ for $0 \le i < m$. Then $\mathcal{B} = \{V_i \mid 0 \le i < m\}$ is a partition of $V\Gamma$. It is easily shown that $V_0 \to V_i, g \mapsto g_i g$ is an isomorphism from $\Gamma[V_0, V_0]$ to $\Gamma[V_i, V_i]$. Thus each subgraph $\Gamma[V_i, V_i]$ contains a spanning subgraph isomorphic to the connected Cayley graph $\Sigma = \mathrm{Cay}(N, Y)$, and so $rc(\Gamma[V_i, V_i]) \le rc(\Sigma)$.

Note that $gNh = ghN$ for $\forall g, h \in G$. Assume that $\Gamma[V_i, V_j]$ is not empty, where $i \ne j$. Then there are some $g, h \in N$ and $x \in S \setminus N$ such that $g_i g x = g_j h$. Thus

$$g_i N x = g_i g N x = g_i g x (x^{-1} N x) = g_i g x N = g_j h N = g_j N.$$

It follows that $\Gamma[V_i, V_j]$ contains a perfect matching, and so $\Gamma[V_i, V_j]$ has no isolated vertices. By Lemma 2.1 (ii), $rc(\Gamma) \le rc(\Sigma) + rc(\Gamma_{\mathcal{B}})$. Consider the quotient graph $\Gamma_{\mathcal{B}}$. Then V_i and V_j are adjacent if and only if $g_j N = g_i N x = (g_i N)(xN)$ for some $x \in S \setminus N$. It follows that $\Gamma_{\mathcal{B}} = \mathrm{Cay}(\bar{G}, \bar{X})$, and hence the result follows. $\square$

Recall that a graph is called *vertex transitive* if for any two vertices there is an automorphism of the graph mapping one vertex to the other one. It is well-known that a connected vertex transitive graph of order no less than three must be 2-connected (see Godsil and Royle 2001, Theorem 3.4.2). Thus, by Li et al. (2012, Theorem 2.4), if Γ is a connected vertex transitive graph then $rc(\Gamma) \le \lceil \frac{|V\Gamma|}{2} \rceil$. Note that a Caley

graph must be vertex-transitive. Then the next two results follow from Theorem 2.2 directly.

Corollary 2.3 *Let Γ, G and N be as in Theorem 2.2. Then $rc(\Gamma) \leq \lceil \frac{|N|}{2} \rceil + \lceil \frac{|G|}{2|N|} \rceil$.*

Corollary 2.4 *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then either*

(i) *G is cyclic and S consists of generators of G; or*
(ii) *there are two proper divisors m and n of $|G|$ such that $|G| = mn$ and $rc(\Gamma) \leq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.*

Proof If $\langle x \rangle = G$ for each $x \in S$, then part (i) follows. Thus we assume that there are $x \in S$ and $Y \subseteq S$ such that $|Y| \geq 1$ and $\langle x, Y \rangle = G$ but $\langle Y \rangle \neq G$. Set $N = \langle Y \rangle$. Then, by Theorem 2.2, $rc(\Gamma) \leq rc(\mathrm{Cay}(G, Y \cup Y^{-1} \cup \{x, x^{-1}\})) \leq rc(\mathrm{Cay}(N, Y \cup Y^{-1})) + rc(\mathrm{Cay}(G/N, \{xN, x^{-1}N\}))$. Note that the Cayley graph $\mathrm{Cay}(G/N, \{xN, x^{-1}N\})$ is either a cycle or the complete graph on two vertices. Then part (ii) follows by setting $|N| = m$ and $|G/N| = n$. □

Now we give a new proof of a known result by using the above simple lemma.

Theorem 2.5 (Li et al. 2011) *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then $rc(\Gamma) \leq \sum_{x \in X} \lceil \frac{|x|}{2} \rceil$, where X is an arbitrary minimal generating set of G contained in S.*

Proof We prove the result by induction on the orders of groups. Let X be an arbitrary minimal generating set of G contained in S. Take $x \in X$, set $Y = X \setminus \{x\}$ and $N = \langle Y \rangle$. Then $G = \langle X \rangle = N \langle x \rangle$, and $|G/N| \leq |\langle x \rangle| = |x|$. By Theorem 2.2, $rc(\mathrm{Cay}(G, X \cup X^{-1})) \leq rc(\mathrm{Cay}(N, Y \cup Y^{-1})) + \lceil \frac{|x|}{2} \rceil$. Since $|N| < |G|$, by induction, we may assume that the result holds for $\mathrm{Cay}(N, Y \cup Y^{-1})$. It is easily shown that Y is also a minimal generating set of N. Thus $rc(\mathrm{Cay}(N, Y \cup Y^{-1})) \leq \sum_{y \in Y} \lceil \frac{|y|}{2} \rceil$, and so $rc(\mathrm{Cay}(G, X \cup X^{-1})) \leq \sum_{x \in X} \lceil \frac{|x|}{2} \rceil$. Then the theorem follows because $rc(\Gamma) \leq rc(\mathrm{Cay}(G, X \cup X^{-1}))$. □

For integers $n \geq 1$ and $m \geq 3$, denote by $\mathbb{Z}_n$ the cyclic group of order n, and by $\mathbf{C}_m$ the cycle of length m. For graphs $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$, the *Cartesian product* $\Sigma_1 \square \Sigma_2 \square \cdots \square \Sigma_r$ is the graph defined on $V\Sigma_1 \times \cdots \times V\Sigma_r$ such that two vertices $(u_1, u_2, \ldots, u_r)$ and $(v_1, v_2, \ldots, v_r)$ are adjacent if and only if there is some $1 \leq i \leq r$ such that $\{u_i, v_i\} \in E\Sigma_i$ and $u_j = v_j$ for all $j \neq i$.

Theorem 2.6 *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then either*

(i) *$rc(\Gamma) < \min\{\sum_{x \in X} \lceil \frac{|x|}{2} \rceil \mid X \subseteq S$ is a minimal generating set of $G\}$; or*
(ii) *$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ and Γ has a connected spanning subgraph $\Sigma_1 \square \Sigma_2 \square \cdots \square \Sigma_r$, where Σ_i is either the n_i-cycle $\mathbf{C}_{n_i}$ if $n_i \geq 3$, or the complete graph K_2 on two vertices if $n_i = 2$.*

Proof By Theorem 2.5, either part (i) follows or there is a minimal generating set $X = \{x_1, x_2, \ldots, x_r\}$ of G such that $rc(\Gamma) = \sum_{i=1}^{r} \lceil \frac{|x_i|}{2} \rceil$, where X is a subset of S. Assume that the latter case occurs. Note that the minimal generating set X is a subset of S. Set $Y = X \setminus \{x_r\}$, $N = \langle Y \rangle$ and $\Sigma = \text{Cay}(N, Y \cup Y^{-1})$. Then Y is a minimal generating set of N. By Theorems 2.2 and 2.5,

$$\sum_{i=1}^{r} \left\lceil \frac{|x_i|}{2} \right\rceil = rc(\Gamma) \leq rc(\Sigma) + \left\lceil \frac{|G|}{2|N|} \right\rceil \leq rc(\Sigma) + \left\lceil \frac{|x_r|}{2} \right\rceil \leq \sum_{i=1}^{r} \left\lceil \frac{|x_i|}{2} \right\rceil.$$

It follows that $rc(\Sigma) = \sum_{i=1}^{r-1} \lceil \frac{|x_i|}{2} \rceil$ and $\lceil \frac{|G|}{2|N|} \rceil = \lceil \frac{|x_r|}{2} \rceil$. Since $G = \langle x_r, Y \rangle = \langle x_r \rangle N$, we have $|G| = |\langle x_r \rangle N| = \frac{|x_r||N|}{|\langle x_r \rangle \cap N|}$, and hence $\frac{|G|}{|N|}$ is a divisor of $|x_r|$. Thus $\lceil \frac{|G|}{2|N|} \rceil = \lceil \frac{|x_r|}{2} \rceil$ implies that either $\frac{|G|}{|N|} = |x_r|$ or $\frac{\frac{|G|}{|N|}+1}{2} = \frac{|x_r|}{2}$. The latter case implies that $\frac{|G|}{|N|}$ and $|x_r|$ are coprime. It follows that $\frac{|G|}{|N|} = 1$ as $\frac{|G|}{|N|}$ is a divisor of $|x_r|$, yielding $G = N$, a contradiction. Therefore, $\frac{|G|}{|N|} = |x_r|$. Thus $|\langle x_r \rangle \cap N| = \frac{|x_r||N|}{|G|} = 1$. Then $G = N\langle x_r \rangle = N \times \langle x_r \rangle$. Since Y is a minimal generating set of N, by induction on r, we may assume that $N = \langle x_1 \rangle \times \cdots \times \langle x_{r-1} \rangle$. Thus $G = \langle x_1 \rangle \times \cdots \times \langle x_{r-1} \rangle \times \langle x_r \rangle \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, where $n_i = |x_i|$ for $1 \leq i \leq r$. Notice that each element of G has the form of $x_1^{e_1} \ldots x_i^{e_i} \ldots x_r^{e_r}$ for integers $e_1, e_2, \ldots, e_r$. Consider the connected spanning subgraph $\text{Cay}(G, X \cup X^{-1})$ of Γ. Then two vertices $x_1^{e_1} \ldots x_i^{e_i} \ldots x_r^{e_r}$ and $x_1^{f_1} \ldots x_i^{f_i} \ldots x_r^{f_r}$ are adjacent in $\text{Cay}(G, X \cup X^{-1})$ if and only if there is some $1 \leq j \leq r$ such that $e_j - f_j \equiv \pm 1 \pmod{n_j}$ and $e_i - f_i \equiv 0 \pmod{n_i}$ for $i \neq j$. Set $\Sigma_i = \text{Cay}(\langle x_i \rangle, \{x_i, x_i^{-1}\})$ for $1 \leq i \leq r$. Then $\text{Cay}(G, X \cup X^{-1}) = \Sigma_1 \square \Sigma_2 \square \cdots \square \Sigma_r$, and so part (ii) follows. $\square$

3 *rc*-numbers of Cayley graphs on dihedral groups

Let $n \geq 1$ be an integer. We use D_{2n} to denote *the dihedral group* generated by two elements, say a and b, such that

$$|a| = n, \ |b| = 2, \ b^{-1}ab = a^{-1}.$$

(Note that $D_2 = \mathbb{Z}_2$ and $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.) Then

$$D_{2n} = \langle a \rangle \cup \langle a \rangle b = \{a^i \mid 0 \leq i < n\} \cup \{a^i b \mid 0 \leq i < n\}.$$

Let $\mathbf{Z}(D_{2n})$ be the center of D_{2n}. Then, for $n \geq 3$, either $\mathbf{Z}(D_{2n}) = 1$, or $\mathbf{Z}(D_{2n}) = \langle a^{\frac{n}{2}} \rangle$ while n is even. For convenience, we collect some basic facts about dihedral groups by considering the involutions, elements of order 2 in D_{2n}.

Lemma 3.1 (i) *For $0 \leq i \leq n-1$, each $a^i b$ is an involution.*
(ii) *If n is odd, then D_{2n} has a unique conjugacy class of involutions, which is $\{a^i b \mid 0 \leq i \leq n-1\}$.*

(iii) *If n is even, then D_{2n} has exactly three conjugacy classes of involutions, which are $\{a^{\frac{n}{2}}\}$, $\{a^{2i}b \mid 0 \le i < \frac{n}{2}\}$ and $\{a^{2i+1}b \mid 0 \le i < \frac{n}{2}\}$.*

(iv) *If m is a divisor of n then $\langle a \rangle$ has a unique subgroup of order m, which is $\langle a^{\frac{n}{m}} \rangle$. If $N \le \langle a \rangle$, then N is normal in D_{2n} and the quotient group D_{2n}/N is a dihedral group generated by $\{aN, bN\}$.*

(v) *If X is a (minimal) generating set of D_{2n}, then X contains some involution $a^s b$, and $(X \cap \langle a \rangle) \cup \{xa^s b \mid a^s b \ne x \in X \setminus \langle a \rangle\}$ is a (minimal) generating set of $\langle a \rangle$.*

(vi) *Set $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ for distinct primes p_i. If Y is a minimal generating set of $\langle a \rangle$, then $|Y| \le r$. If X is a minimal generating set of D_{2n}, then $|X| \le r + 1$.*

Proof Items (i)–(iv) can be found in most text books about elementary group theory. Here we only prove items (v) and (vi).

Let X be an arbitrary generating set of D_{2n}. Then $\langle a \rangle \ne D_{2n} = \langle X \rangle$, yielding $a^s b \in X$ for some integer s. Set $Y = (X \cap \langle a \rangle) \cup \{xa^s b \mid a^s b \ne x \in X \setminus \langle a \rangle\}$. Then $Y \subseteq \langle a \rangle$ as $xa^s b \in \langle a \rangle$ for $x \in X \setminus \langle a \rangle$, and so $\langle Y \rangle$ is normal in D_{2n}. Thus

$$D_{2n} = \langle X \rangle = \langle X \cap \langle a \rangle, X \setminus \langle a \rangle \rangle = \langle Y, a^s b \rangle = \langle Y \rangle \langle a^s b \rangle.$$

It follows that $\langle a \rangle = \langle Y \rangle$. Assume that Y is not a minimal generating set of $\langle a \rangle$. Then $\langle Y \setminus \{y\} \rangle = \langle a \rangle$ for some $y \in Y$. If $y \in X \cap \langle a \rangle$ then $D_{2n} = \langle X \setminus \{y\} \rangle$, and so X is not a minimal generating set of D_{2n}. Suppose that $y = za^s b$ for some $z \in X \setminus \langle a \rangle$ with $z \ne a^s b$. Set $Y_1 = Y \setminus \{za^s b\}$. Then

$$\langle X \setminus \{z\} \rangle = \langle X \cap \langle a \rangle, X \setminus (\langle a \rangle \cup \{z, a^s b\}), a^s b \rangle = \langle Y_1 \rangle \langle a^s b \rangle = \langle Y \rangle \langle a^s b \rangle = D_{2n}.$$

It implies that X is not a minimal generating set of D_{2n}. Thus item (v) holds.

Now set $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ for distinct primes p_i. For $1 \le i \le r$, set $a_i = a^{\prod_{j \ne i} p_j^{e_j}}$. Then $\langle a \rangle = \langle a_i \mid 1 \le i \le r \rangle = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle$, and so each element in $\langle a \rangle$ can be written as a product of the form of $\prod_{i=1}^{r} a_i^{m_i}$. Let $Y = \{y_j \mid 1 \le j \le t\}$ be an arbitrary generating set of $\langle a \rangle$. Write $y_j = \prod_{i=1}^{r} a_i^{f_i(y_j)}$, where all f_i are integral valued functions on Y. Then

$$\langle a \rangle = \langle Y \rangle = \langle a_i^{f_i(y_j)} \mid 1 \le i \le r, 1 \le j \le t \rangle.$$

For each $1 \le j \le r$, we choose one element $y \in Y$ such that $|a_j^{f_j(y)}| = \max\{|a_j^{f_j(y_i)}| \mid 1 \le i \le t\}$. Thus we obtain r elements contained in Y which are not necessarily distinct, say $y_1', y_2', \ldots, y_r'$. Set $Y_1 = \{y_j' \mid 1 \le j \le r\}$. Then

$$\langle Y_1 \rangle = \langle a_i^{f_i(y_j')} \mid 1 \le i, j \le r \rangle.$$

By item (iv) and the choice of elements $a_j^{f_j(y_j')}$, we have

$$\langle a_i^{f_i(y_j')} \mid 1 \le i, j \le r \rangle = \langle a_j^{f_j(y_j')} \mid 1 \le j \le r \rangle = \langle a_i^{f_i(y_j)} \mid 1 \le i \le r, 1 \le j \le t \rangle.$$

It follows that $\langle Y_1 \rangle = \langle Y \rangle = \langle a \rangle$. Thus each generating set of $\langle a \rangle$ contains a generating set of size no more than r. Then the first part of item (vi) follows, while the second part follows from item (v). □

It is well-known and easily shown that the ladder graph L_n and Möbius ladder graph M_n (of order $2n$) are cubic Cayley graphs on the dihedral group D_{2n}. Cai et al. (to appear) proved that

$$rc(L_n) = \left\lceil \frac{n+1}{2} \right\rceil \quad \text{and} \quad rc(M_n) = \left\lceil \frac{n}{2} \right\rceil.$$

In this section we investigate the rc-numbers of the whole class of connected Cayley graphs on dihedral groups. By Theorem 2.2, the following lemma holds.

Lemma 3.2 *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected Cayley graph. Then*

$$rc(\Gamma) \le rc(\mathrm{Cay}(\langle S_1 \rangle, S_1)) + rc(\mathrm{Cay}(D_{2m}, \bar{S})),$$

where $S_1 = \langle a \rangle \cap S$, $m = \frac{n}{|\langle S_1 \rangle|}$, $\bar{S}$ is a set of involutions in D_{2m} and either $m \le 2$ or $\bar{S} \cap \mathbf{Z}(D_{2m}) = \emptyset$.

Note that, in the above lemma, $\mathrm{Cay}(\langle S_1 \rangle, S_1)$ is a Cayley graph on a cyclic group. Thus, in view of Theorem 2.6, we next consider the Cayley graphs $\mathrm{Cay}(D_{2n}, S)$ under the assumption that $S \cap \langle a \rangle = \emptyset$.

Theorem 3.3 *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected Cayley graph. Assume that $S = \{a^{e_i} b \mid 1 \le i \le r\}$ for integers $e_1, e_2, \ldots, e_r$. Then one of the following holds.*

 (i) *Γ is a cycle;*
 (ii) *$r \ge 3$, n is odd and $e_i - e_j$ are coprime to n, where $i \ne j$ and $1 \le i, j \le r$;*
 (iii) *$rc(\Gamma) \le (rc(\mathrm{Cay}(D_{2l}, X)) + 1)\lceil \frac{m}{2} \rceil$ for some divisors l and m of n with $l, m \ge 2$ and $n = lm$, where X is a set of involutions in D_{2l} such that $|X| \le |S| - 1$ and either $l = 2$ or $X \cap \mathbf{Z}(D_{2l}) = \emptyset$.*

Proof We assume first that D_{2n} is generated by any 2-subset of S. If $r = 2$ then part (i) follows. Suppose that $r \ge 3$. By the assumption, for distinct i and j,

$$D_{2n} = \langle a^{e_i} b, a^{e_j} b \rangle = \langle a^{e_i - e_j}, a^{e_j} b \rangle = \langle a^{e_i - e_j} \rangle \langle a^{e_j} b \rangle.$$

It follows that $\langle a \rangle = \langle a^{e_i - e_j} \rangle$, which yields that $e_i - e_j$ is coprime to n. Since $r \ge 3$, take $k \ne i, j$. Then $e_i - e_j$, $e_i - e_k$ and $e_j - e_k$ are coprime to n. Note that one of $e_i - e_j$, $e_i - e_k$ and $e_j - e_k$ must be even. It implies that n is odd.

Let $G = D_{2n}$. Now we may assume that there are $x \in S$ and $Y \subseteq S$ such that $|Y| \ge 2$ and $\langle x, Y \rangle = G$ but $\langle Y \rangle \ne G$. Without loss of generality, we may set $x = a^{e_r} b$ and $Y = \{a^{e_j} b \mid 1 \le j \le s\}$, where $2 \le s \le r - 1$. Let $H = \langle Y \rangle$. Then $H = \langle Y \rangle = \langle Y a^{e_s} b, a^{e_s} b \rangle$. Observe that $Y a^{e_s} b \setminus \{1\} = \{a^{e_j - e_s} \mid 1 \le j \le s - 1\} \subset \langle a \rangle$, we conclude that $H = \langle Y \rangle = L \langle a^{e_s} b \rangle$, where $L = \langle a^e \rangle$ for some integer e. Clearly, $|H| \ge 4$, $l := |a^e| = |L| = \frac{|H|}{2} \ge 2$ and $m := \frac{|G|}{|H|} = \frac{n}{l} \ge 2$. Moreover,

$a^m \in L = \langle a^e \rangle$ and H has m left cosets in G. Note that $G = \langle a, b \rangle = \langle a, a^{es}b \rangle = \langle a \rangle \langle a^{es}b \rangle$. Then each left coset of H has the form of $a^j H$ for some $0 \le j < m$. Notice that

$$G = \langle x, Y \rangle = \langle x, H \rangle = \langle a^{er}b, a^e, a^{es}b \rangle = \langle a^e, a^{es-er} \rangle \langle x \rangle = \langle a^e, a^{er-es} \rangle \langle x \rangle.$$

It follows that $\langle a \rangle = \langle a^e, a^{er-es} \rangle$. Thus for each $0 \le j < m$, there are some integers j_1 and j_2 such that $a^j = (a^{er-es})^{j_1}(a^e)^{j_2}$, and so $a^j H = a^{j_1(er-es)}H$. Recalling $a^m \in \langle a^e \rangle \le H$, we have $a^{m(er-es)} \in H$. It follows that each left coset of H has the form of $a^{i(er-es)}H$ for some $0 \le i < m$. Set $V_i = a^{i(er-es)}H$ for $0 \le i < m$. Then each subgraph $\Gamma[V_i, V_i]$ contains a spanning subgraph isomorphic to the connected Cayley graph $\mathrm{Cay}(H, Y)$.

Note that

$$H = \langle a^e \rangle \cup \langle a^e \rangle a^{es}b = \langle a^e \rangle \cup \langle a^e \rangle a^{es-er}a^{er}b = \langle a^e \rangle \cup a^{es-er}\langle a^e \rangle x.$$

Then

$$V_i = a^{i(er-es)}H = a^{i(er-es)}\langle a^e \rangle \cup a^{(i-1)(er-es)}\langle a^e \rangle x.$$

Set $V_i^1 = a^{i(er-es)}\langle a^e \rangle$ and $V_i^2 = a^{(i-1)(er-es)}\langle a^e \rangle x$. Then $\Gamma[V_i^1, V_{i+1}^2]$ contains a perfect matching (on $V_i^1 \cup V_{i+1}^2$), and $\Gamma[V_{i-1}^1, V_i^2]$ contains a perfect matching (on $V_{i-1}^1 \cup V_i^2$). Thus, noting that $H \cong D_{2l}$, the result follows from Lemma 2.1 (i). □

Recall that a cycle of length $2m$ has rc-number m. Then we get the following consequence of Theorem 3.3 by induction on the sizes of minimal generating sets of D_{2n}.

Corollary 3.4 *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected Cayley graph with n even. Assume that $S = \{a^{e_i}b \mid 1 \le i \le r\}$ is a minimal generating set of D_{2n}. Then either Γ is a cycle of length $2n$, or*

$$rc(\Gamma) \le m_1 \prod_{j=2}^{r-1} \left\lceil \frac{m_j}{2} \right\rceil + \sum_{i=2}^{r-1} \prod_{j=i}^{r-1} \left\lceil \frac{m_j}{2} \right\rceil$$

for some divisors m_j of n with $n = \prod_{j=1}^{r-1} m_j$ and $m_j \ge 2$ for $1 \le j \le r-1$.

In the next result we investigate the rc-numbers of cubic Cayley graphs on dihedral groups.

Theorem 3.5 *Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected cubic Cayley graph. Then one of the following cases occurs.*

(i) $rc(\Gamma) = \lceil \frac{n+1}{2} \rceil$, *and Γ is the ladder graph of order $2n$.*
(ii) $rc(\Gamma) = \lceil \frac{n}{2} \rceil$, *and Γ is the Möbius ladder of order $2n$.*
(iii) $\Gamma \cong \mathrm{Cay}(D_{2n}, \{b, a^s b, a^t b\})$ *for some integers s and t, and either*

(iii.1) $rc(\Gamma) \le (l+1)\lceil\frac{m}{2}\rceil$, where $l \in \{|a^s|, |a^t|, |a^{s-t}|\}$ and $m = \frac{n}{l} \ge 2$; or

(iii.2) n is odd, and s, t and $s - t$ are coprime to n.

Proof Since $|S| = 3$ and $S^{-1} = S$, we conclude that S contains an involution z of D_{2n}. Set $S = \{x, y, z\}$. Then either $x = y^{-1}$ has order no less than 3, or x, y and z are distinct involutions. Thus one of the following cases occurs:

(1) $x = y^{-1}$ has order no less than 3;

(2) n is even, x, y and z are involutions, and $a^{\frac{n}{2}} \in S$.

(3) $x, y, z \in \{a^i b \mid 0 \le i \le n - 1\}$.

We next prove this theorem by dealing with these three cases separately.

Assume that (1) holds. Since Γ is connected, $\langle S \rangle = D_{2n}$. It follows from the assumption that $x = a^i$ and $z = a^j b$ for some integers i and j. Then $zxz = x^{-1}$ and $D_{2n} = \langle x, z \rangle = \langle x \rangle \langle z \rangle$. Thus $2n = |D_{2n}| = |\langle x \rangle \langle z \rangle| = |x||z| = 2|x|$, yielding $|x| = n$. Then Γ has two cycles of length n, say $(1, x, x^2, \ldots, x^k, \ldots, x^{n-1}, 1)$ and $(z, zx, \ldots, zx^k, \ldots, zx^{n-1}, z)$. Moreover, Γ is constructed from these two cycles by adding a perfect matching $\{\{x^i, zx^{n-i}\} \mid 0 \le i < n\}$. This says that Γ is (isomorphic to) the ladder graph of order $2n$. By Cai et al. (to appear), $rc(\Gamma) = \lceil\frac{n+1}{2}\rceil$, and so part (i) of this theorem occurs.

Assume that (2) holds. Without loss of generality, we may set $z = a^{\frac{n}{2}}$ and write $x = a^s b$ and $y = a^t b$ for some integers s and t. Thus $xy = a^{s-t}$ and $\langle x, y \rangle = \langle xy \rangle \langle x \rangle$.

Suppose that $\langle x, y \rangle$ is a proper subgroup of D_{2n}. Since Γ is connected, $D_{2n} = \langle x, y, z \rangle$. Note that $z = a^{\frac{n}{2}}$ lies in the center of D_{2n}. Thus $D_{2n} = \langle x, y, z \rangle = \langle z \rangle \langle x, y \rangle \ne \langle x, y \rangle$. It implies that $|\langle x, y \rangle| = n$. For $0 \le i < \frac{n}{2}$, set $u_{2i} = (xy)^i$, $u_{2i+1} = (xy)^i x$, $w_{2i} = (xy)^i z$ and $w_{2i+1} = (xy)^i xz$. Then Γ is constructed from two cycles $(u_0, u_1, \ldots, u_{n-1}, u_0)$ and $(w_0, w_1, \ldots, w_{n-1}, w_0)$ by adding a perfect matching $\{\{u_i, w_i\} \mid 0 \le i < n\}$. Thus Γ is the the ladder graph of order $2n$ and, by Cai et al. (to appear), part (i) of this theorem occurs.

Suppose that $D_{2n} = \langle x, y \rangle$. Then $|xy| = n$, and so $s - t$ is coprime to n; in particular, $s - t$ is odd. It follows that $z = a^{\frac{n}{2}} = (a^{s-t})^{\frac{n}{2}} = (xy)^{\frac{n}{2}}$. Set $v_{2i} = (xy)^i$ and $v_{2i+1} = (xy)^i x$ for $0 \le i < n$. Then Γ has a hamiltonian cycle $(v_0, v_1, v_2, \ldots, v_{2n-1}, v_0)$ and a perfect matching $\{\{v_i, v_{i+n}\} \mid 0 \le i < n\}$. Hence Γ is (isomorphic to) the Möbius ladder of order $2n$. By Cai et al. (to appear), part (ii) of this theorem occurs.

Now we deal with case (3). By Lemma 3.1(ii) and (iii), one of x, y and z is conjugate to b. Without loss of generality, we assume that $b = g^{-1}zg$ for some $g \in D_{2n}$. Write $g^{-1}xg = a^s b$ and $g^{-1}yg = a^t b$. Set $T = \{b, a^s b, a^t b\}$ and $\Sigma = \text{Cay}(D_{2n}, T)$. It is easily shown that $V\Gamma \to V\Sigma, h \mapsto g^{-1}hg$ is an isomorphism from Γ to Σ. Note that two isomorphic graphs have the same rc-number. Thus $rc(\Gamma) = rc(\Sigma)$, and part (iii) of this theorem follows from Theorem 3.3 and Li et al. (2012, Theorem 2.4). $\square$

We end this section by considering the rc-numbers of Cayley graphs on $G = D_{2p^k}$ or D_{2pq}, where $k \ge 1$ is an integer, p and q are distinct primes.

Let X be a minimal generating set of G, $S = X \cup X^{-1}$ and $\Gamma = \text{Cay}(G, S)$. Then $2 \le |X| \le 3$ by Lemma 3.1 (vi). Suppose that $|X| = 2$. Then either $X = \{a^i b, a^j b\}$ or $X = \{a^i, a^j b\}$ for some integers i and j. Thus $S = X$ or $S = \{a^i, a^{-i}, a^j b\}$. It follows that Γ is either a cycle or a ladder graph.

Therefore, we assume next that $|X| = 3$. By Lemma 3.1 (v), $\langle a \rangle$ has a minimal generating set of size 2. It follows that $|a| = |\langle a \rangle|$ is not a power of some prime. Thus $|a| = pq$ and $G = D_{2pq}$. By Lemma 3.1 (v), we have $a^s b \in X$ for some integer s. Suppose that $\langle a \rangle \cap X = \emptyset$. Then $X = \{a^r b, a^t b, a^s b\}$ with $\{|a^{r-s}|, |a^{t-s}|\} = \{p, q\}$, where r and t are integers. It follows that $a^{r-t} = a^{r-s} a^{-(t-s)}$ has order pq, and hence $\langle a^r b, a^t b \rangle = D_{2pq}$, a contradiction. Thus $\langle a \rangle \cap X \neq \emptyset$, and hence one of the following cases occurs:

I. $X = \{a^r, a^t, a^s b\}$ with $\{|a^r|, |a^t|\} = \{p, q\}$.
II. $X = \{a^r, a^t b, a^s b\}$ with $\{|a^r|, |a^{t-s}|\} = \{p, q\}$.

For case I, by Theorems 2.2 and 2.5, we get $rc(\Gamma) \leq \lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil + 1$. In the following we discuss case II.

Assume case II occurs. Set $|a^r| = l$, $|a^{t-s}| = m$ and $H = \langle a^t b, a^s b \rangle$. Then $H \cong D_{2m}$, $D_{2pq} = \langle a^r \rangle H$ and $\langle a^r \rangle \cap H = 1$. In particular, every element of D_{2pq} has the form of $a^{ir} h$, where $0 \leq i < l$ and $h \in H$. Thus it is easily shown that $\Gamma = \Sigma_1 \square \Sigma_2$, where $\Sigma_1 = \mathrm{Cay}(\langle a^r \rangle, \{a^r, a^{-r}\}) \cong K_2$ or C_l, and $\Sigma_2 = \mathrm{Cay}(H, \{a^s b, a^t b\}) \cong C_{2m}$. Suppose $\Sigma_1 \cong K_2$. Then Γ is isomorphic to the ladder graph L_{2m}. Suppose $\Sigma_1 \cong C_l$. By Liang (2012), $rc(C_l \square C_{2m}) = \frac{l-1}{2} + m$. Therefore, $rc(\Gamma) = \frac{l-1}{2} + m$ with $\{l, m\} = \{p, q\}$.

By the foregoing argument we obtain the following result.

Theorem 3.6 *Let $G = D_{2p^k}$ or D_{2pq}, where $k \geq 1$ is an integer, p and q are distinct primes. Let X be a minimal generating set of G. Set $S = X \cup X^{-1}$ and $\Gamma = \mathrm{Cay}(G, S)$. Then one of the following statements holds.*

(i) *Γ is either a cycle or a ladder graph.*
(ii) *$G = D_{2pq}$, $|X| = 3$ and either*
(ii.1) *$|\langle a \rangle \cap X| = 2$ and $rc(\Gamma) \leq \lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil + 1$; or*
(ii.2) *$|\langle a \rangle \cap X| = 1$ and $rc(\Gamma) = \lfloor \frac{l}{2} \rfloor + m$ with $\{l, m\} = \{p, q\}$.*

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