

3 RAINBOW VERTEX-CONNECTION AND FORBIDDEN SUBGRAPHS

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20 **Abstract**

21 A path in a vertex-colored graph is called *vertex-rainbow* if its internal
22 vertices have pairwise distinct colors. A vertex-colored graph G is *rainbow*
23 *vertex-connected* if for any two distinct vertices of G , there is a vertex-
24 rainbow path connecting them. For a connected graph G , the *rainbow vertex-*
25 *connection number* of G , denoted by $rvc(G)$, is defined as the minimum
26 number of colors that are required to make G rainbow vertex-connected. In
27 this paper, we find all the families \mathcal{F} of connected graphs with $|\mathcal{F}| \in \{1, 2\}$,
28 for which there is a constant $k_{\mathcal{F}}$ such that, for every connected \mathcal{F} -free graph
29 G , $rvc(G) \leq diam(G) + k_{\mathcal{F}}$, where $diam(G)$ is the diameter of G .

30 **Keywords:** vertex-rainbow path; rainbow vertex-connection; forbidden sub-
31 graphs.

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34

1. INTRODUCTION

35 All graphs considered in this paper are simple, finite, and undirected. We
 36 follow the terminology and notation of Bondy and Murty in [2] for those not
 37 defined here.

38 Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \rightarrow$
 39 $\{0, 1, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color.
 40 A path in G is called a *rainbow path* if no two edges of the path are colored
 41 with the same color. The graph G is called *rainbow connected* if for any two
 42 distinct vertices of G , there is a rainbow path connecting them. For a connected
 43 edge-colored graph G , the *rainbow connection number* of G , denoted by $rc(G)$,
 44 is defined as the minimum number of colors that are needed to make G rainbow
 45 connected. Observe that if G has n vertices, then $diam(G) \leq rc(G) \leq n - 1$.
 46 It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, and
 47 $rc(G) = n - 1$ if and only if G is a tree. The concept of rainbow connection of
 48 graphs was first introduced by Chartrand et al. in [3], and has been well-studied
 49 since then. For further details, we refer the reader to a survey paper [10] and a
 50 book [11].

51 Let G be a nontrivial connected graph with a *vertex-coloring* $c : V(G) \rightarrow$
 52 $\{0, 1, \dots, t\}$, $t \in \mathbb{N}$, where adjacent vertices may be colored with the same col-
 53 or. A path of G is called *vertex-rainbow* if any two internal vertices of the path
 54 have distinct colors. The vertex-colored graph G is *rainbow vertex-connected* if
 55 any two vertices of G are connected by a vertex-rainbow path. For a connected
 56 graph G , the *rainbow vertex-connection number* of G , denoted by $rvc(G)$, is the
 57 minimum number of colors used in a vertex-coloring of G to make G rainbow
 58 vertex-connected. The concept of rainbow vertex-connection of graphs was pro-
 59 posed by Krivelevich and Yuster in [6]. They showed that if G is a connected
 60 graph with n vertices and minimum degree δ , then $rvc(G) \leq 11n/\delta$. In [9], Li
 61 and Shi improved this bound. In [4], it was shown that computing the rainbow
 62 vertex-connection number of a graph is NP-hard. Recently, Li et al. in [7] proved
 63 that it is NP-complete to decide whether a given vertex-colored graph is rainbow
 64 vertex-connected even when the graph is bipartite.

65 For the rainbow vertex-connection number of graphs, the following observa-
 66 tions are immediate.

67 **Proposition 1.** Let G be a connected graph with n vertices. Then

- 68 (i) $diam(G) - 1 \leq rvc(G) \leq n - 2$;
 69 (ii) $rvc(G) = diam(G) - 1$ if $diam(G) = 1$ or 2 , with the assumption that
 70 complete graphs have rainbow vertex-connection number 0 .

71 Note that the difference $rvc(G) - diam(G)$ can be arbitrarily large. In fact, if
 72 G is a subdivision of a star $K_{1,n}$, then we have $rvc(G) - diam(G) = (n + 1) - 4 =$

73 $n - 3$, since in a rainbow vertex-connected coloring of G , the internal vertices
 74 must have distinct colors.

75 In [8], Li and Liu studied the rainbow vertex-connection number for any
 76 2-connected graph, and determined the precise value of the rainbow vertex-
 77 connection number of the cycle C_n ($n \geq 3$).

Theorem 1. [8] Let C_n be a cycle of order n ($n \geq 3$). Then,

$$rvc(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 4, 5; \\ 3 & \text{if } n = 9; \\ \lfloor \frac{n}{2} \rfloor - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13, \text{ or } 15; \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \geq 16 \text{ or } n = 14. \end{cases}$$

78 Let \mathcal{F} be a family of connected graphs. We say that a graph G is \mathcal{F} -free
 79 if G does not contain any induced subgraph isomorphic to a graph from \mathcal{F} .
 80 Specifically, for $\mathcal{F} = \{X\}$ we say that G is X -free, and for $\mathcal{F} = \{X, Y\}$ we say
 81 that G is (X, Y) -free. The members of \mathcal{F} will be referred to in this context as
 82 *forbidden induced subgraphs*, and for $|\mathcal{F}| = 2$ we also say that \mathcal{F} is a *forbidden*
 83 *pair*.

84 In [5], Holub et al. considered the question: For which families \mathcal{F} of connected
 85 graphs, a connected \mathcal{F} -free graph G satisfies $rc(G) \leq diam(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is
 86 a constant (depending on \mathcal{F})? They gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in
 87 the following two results (where N denotes the *net*, a graph obtained by attaching
 88 a pendant edge to each vertex of a triangle).

89 **Theorem 2.** [5] Let X be a connected graph. Then there is a constant k_X such
 90 that every connected X -free graph G satisfies $rc(G) \leq diam(G) + k_X$, if and only
 91 if $X = P_3$.

92 **Theorem 3.** [5] Let X, Y be connected graphs such that $X, Y \neq P_3$. Then
 93 there is a constant k_{XY} such that every connected (X, Y) -free graph G satisfies
 94 $rc(G) \leq diam(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ ($r \geq 4$)
 95 and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N .

96 Naturally, we may consider an analogous question concerning the rainbow
 97 vertex-connection number of graphs. In this paper, we will consider the following
 98 question.

99 *For which families \mathcal{F} of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a
 100 connected graph G being \mathcal{F} -free implies $rvc(G) \leq diam(G) + k_{\mathcal{F}}$?*

101 We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and for $|\mathcal{F}| = 2$ in Section
 102 4.

103

2. PRELIMINARIES

104 In this section, we introduce some further notations and facts that will be
 105 needed for the proofs of our main results.

106 If G is a graph and $A \subset V(G)$, then $G[A]$ denotes the subgraph of G induced
 107 by the vertex set A , and $G - A$ the graph $G[V(G) \setminus A]$. An edge is called a
 108 *pendant edge* if one of its end vertices has degree one. The *subdivision* of a graph
 109 G is the graph obtained from G by adding a vertex of degree 2 to each edge of G .
 110 For $x, y \in V(G)$, a path in G from x to y will be referred to as an (x, y) -*path*, and,
 111 whenever necessary, it will be considered as oriented from x to y . For a subpath
 112 of a path P with origin u and terminus v (also referred to as a (u, v) -*arc* of P),
 113 we will use the notation uPv . If w is a vertex of a path with a fixed orientation,
 114 then w^- and w^+ denote the predecessor and successor of w , respectively.

115 For graphs X and G , we write $X \subset G$ if X is a subgraph of G , $X \overset{\text{IND}}{\subset} G$ if
 116 X is an induced subgraph of G , and $X \simeq G$ if X is isomorphic to G . For two
 117 vertices $x, y \in V(G)$, we use $\text{dist}_G(x, y)$ to denote the distance between x and
 118 y in G . The diameter of G is defined as the maximum of $\text{dist}_G(x, y)$ among all
 119 pairs of vertices x, y of G , and will be denoted by $\text{diam}(G)$. A shortest path
 120 joining two vertices at distance $\text{diam}(G)$ will be referred to as a *diameter path*.
 121 The *distance between a vertex $u \in V(G)$ and a set $S \subset V(G)$* is defined as
 122 $\text{dist}_G(u, S) := \min_{v \in S} \text{dist}_G(u, v)$. A set $D \subset V(G)$ is called *dominating* if every
 123 vertex in $V(G) \setminus D$ has a neighbor in D . In addition, if $G[D]$ is connected, then
 124 we call D a *connected dominating set*. Throughout this paper, \mathbb{N} denotes the set
 125 of all positive integers.

126 For a set $S \subset V(G)$ and $k \in \mathbb{N}$, the *kth-neighborhood* of S is the set $N_G^k(S)$ of
 127 all vertices of G at distance k from S . In the special case $k = 1$, we simply write
 128 $N_G(S)$ for $N_G^1(S)$, and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For
 129 a set $M \subset V(G)$, we denote $N_M^k(S) = N_G^k(S) \cap M$ and $N_M^k(x) = N_G^k(x) \cap M$,
 130 and as above, we simply use $N_M(S)$ for $N_M^1(S)$ and $N_M(x)$ for $N_M^1(x)$. For a
 131 subgraph $P \subset G$, we write $N_P(x)$ for $N_{V(P)}(x)$. Finally, we will use P_k to denote
 132 the path on k vertices.

133 We end up this section with an important result that will be used in our
 134 proofs.

135 **Theorem 4.** [1] Let G be a connected P_5 -free graph. Then G has a dominating
 136 clique or a dominating P_3 .

137

3. FAMILIES WITH ONE FORBIDDEN SUBGRAPH

138 In this section, we characterize all connected graphs X such that every con-
 139 nected X -free graph G satisfies $\text{rvc}(G) \leq \text{diam}(G) + k_X$, where k_X is a constant.

140 **Theorem 5.** Let X be a connected graph. Then there is a constant k_X such
 141 that every connected X -free graph G satisfies $rvc(G) \leq diam(G) + k_X$, if and
 142 only if $X = P_3$ or P_4 .

143 **Proof.** We have $diam(G) \leq 2$ since G is P_4 -free. Then it follows from Proposi-
 144 tion 1 that $rvc(G) = diam(G) - 1 \leq 1$.

145 Conversely, let $t \geq k_X + 5$, and G_1^t be the subdivision of $K_{1,t}$, and let G_2^t
 146 denote the graph obtained by attaching a pendant edge to each vertex of the
 147 complete graph K_t (see Fig.1). Since $rvc(G_1^t) = t + 1$ but $diam(G_1^t) = 4$, X is
 148 an induced subgraph of G_1^t . Clearly, $rvc(G_2^t) = t$ but $diam(G_2^t) = 3$, and G_2^t is
 149 $K_{1,3}$ -free and P_5 -free. Hence, X is an induced subgraph of P_4 .

150 The proof is thus complete. ■



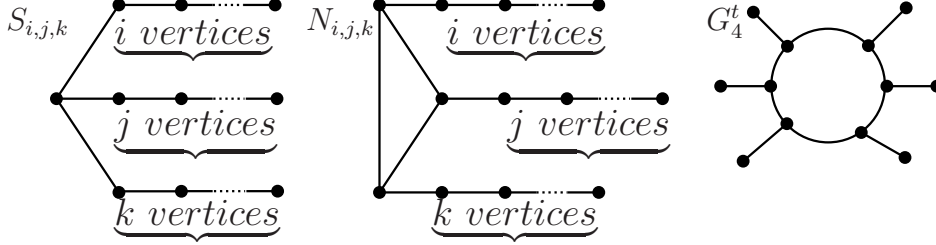
Figure 1: The graphs G_1^t and G_2^t .

151 4. FAMILIES WITH A PAIR OF FORBIDDEN SUBGRAPHS

152 For $i, j, k \in \mathbb{N}$, let $S_{i,j,k}$ denote the graph obtained by identifying one end-
 153 vertex from each of three vertex-disjoint paths of lengths i, j, k , and $N_{i,j,k}$ denote
 154 the graph obtained by identifying each vertex of a triangle with an endvertex of
 155 one of three vertex-disjoint paths of lengths i, j, k (see Fig.2). In this context, we
 156 will also write K_t^h for the graph G_2^t introduced in the proof of Theorem 5.

157 The following statement, which is the main result of this section, characterizes
 158 all forbidden pairs X, Y for which there is a constant k_{XY} such that G being
 159 (X, Y) -free implies $rvc(G) \leq diam(G) + k_{XY}$. By virtue of Theorem 5, we
 160 exclude the case that one of X, Y is an induced subgraph of P_4 . Recall that the
 161 *net* is the graph $N = N_{1,1,1}$.

162 **Theorem 6.** Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs. Then there is
 163 a constant k_{XY} such that every connected (X, Y) -free graph G satisfies $rvc(G) \leq$
 164 $diam(G) + k_{XY}$, if and only if (up to symmetry) $X = P_5$ and $Y \overset{\text{IND}}{\subset} K_r^h$ ($r \geq 4$),
 165 or $X \overset{\text{IND}}{\subset} S_{1,2,2}$ and $Y \overset{\text{IND}}{\subset} N$.

Figure 2: The graphs $S_{i,j,k}$, $N_{i,j,k}$ and G_4^t .

166 The proof of Theorem 6 will be divided into three separate results: we prove
 167 the necessity in Proposition 2, and Theorems 7 and 8 will establish the sufficiency
 168 of the forbidden pairs given in Theorem 6.

169 **Proposition 2.** Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs for which
 170 there is a constant k_{XY} such that every connected (X, Y) -free graph G satisfies
 171 $rvc(G) \leq diam(G) + k_{XY}$. Then, (up to symmetry) $X = P_5$ and $Y \stackrel{\text{IND}}{\subset} K_r^h$ ($r \geq 4$),
 172 or $X \stackrel{\text{IND}}{\subset} S_{1,2,2}$ and $Y \stackrel{\text{IND}}{\subset} N$.

173 **Proof.** Let $t \geq 2k_{XY} + 5$, and let (see Fig.2):

- 174 • $G_3^t = N_{t-1, t-1, t-1}$;
- 175 • G_4^t be the graph obtained by attaching a pendant edge to each vertex of a
 176 cycle C_t .

177 We will also use the graphs G_1^t and $G_2^t (= K_t^h)$ shown in Fig.1.

178 For the graphs G_1^t and G_2^t , we have $diam(G_1^t) = 4$ but $rvc(G_1^t) = t + 1$, and
 179 $diam(G_2^t) = 3$ but $rvc(G_2^t) = t$, respectively. For the graph G_3^t , we observe that
 180 $diam(G_3^t) = 2t - 1$ while $rvc(G_3^t) = 3(t - 1) = \frac{3}{2}(diam(G_3^t) - 1)$, since all internal
 181 vertices must have mutually distinct colors. Analogously, for the graph G_4^t , we
 182 have $diam(G_4^t) = \lfloor \frac{t}{2} \rfloor + 2$, but $rvc(G_4^t) = t \geq 2(diam(G_4^t) - 2)$. Thus, each of the
 183 graphs G_1^t, G_2^t, G_3^t and G_4^t must contain an induced subgraph isomorphic to one
 184 of the graphs X, Y .

185 Consider the graph G_1^t . Up to symmetry, we have that X is an induced
 186 subgraph of G_1^t excluding P_3 and P_4 . Now we consider the graph G_2^t . Obviously,
 187 G_2^t is X -free since G_2^t is $K_{1,3}$ -free. Hence, G_2^t contains Y , implying $Y \stackrel{\text{IND}}{\subset} K_r^h$ for
 188 some $r \geq 3$ (for $r \leq 2$ we get $Y \stackrel{\text{IND}}{\subset} P_4$, which is excluded by the assumptions).

189 Now we consider the graph G_3^t . There are two possibilities:

- 190 (i) $Y \stackrel{\text{IND}}{\subset} G_3^t$. Then $Y \stackrel{\text{IND}}{\subset} N$. Now we consider the graph G_4^t . G_4^t is N -free,
 191 so we get $X \stackrel{\text{IND}}{\subset} S_{1,2,2}$.

192 (ii) $X \stackrel{\text{IND}}{\subset} G_3^t$. Then $X = P_5$. As the case $X = P_5$ and $Y = N$ is already
 193 covered by case (i), we have that $X = P_5$ and $Y \stackrel{\text{IND}}{\subset} K_r^h$, $r \geq 4$.

194 This completes the proof. \blacksquare

195 It is easy to observe that if $X \stackrel{\text{IND}}{\subset} X'$, then every (X, Y) -free graph is also
 196 (X', Y) -free. Thus, when proving the sufficiency of Theorem 6, we will be always
 197 interested in *maximal pairs* of forbidden subgraphs, i.e., pairs X, Y such that, if
 198 replacing one of X, Y , say X , with a graph $X' \neq X$ such that $X \stackrel{\text{IND}}{\subset} X'$, then the
 199 statement under consideration is not true for (X', Y) -free graphs.

200 **Theorem 7.** Let G be a connected (P_5, K_r^h) -free graph for some $r \geq 4$. Then,
 201 $\text{rvc}(G) \leq \text{diam}(G) + r$.

202 **Proof.** From Theorem 4, we have that G has a dominating clique or a dominating
 203 P_3 .

204 **Case 1.** G has a dominating P_3 .

205 We color the vertices of P_3 with colors 1, 2, 3 and color the remaining vertices
 206 arbitrarily (e.g., all of them have color 1). One can easily check that this vertex-
 207 coloring can make G rainbow vertex-connected. So, in this case, $\text{rvc}(G) \leq 3 \leq$
 208 $\text{diam}(G) + r$.

209 **Case 2.** G has a dominating clique, denoted by K_p .

210 Set $W = V(G) \setminus V(K_p)$, $H = G \setminus E(K_p)$. Let A be an independent set in
 211 $G[W]$ and $B \subset V(K_p)$ such that $H[A \cup B] = \ell K_2$ (that is, a matching of order ℓ)
 212 and ℓ is maximal. Then $\ell < r$, for otherwise, $G[A \cup B]$ contains an induced K_r^h .
 213 Moreover, for $x \in W \setminus A$, $N_{A \cup B}(x) \neq \emptyset$, since ℓ is maximal. Now we define the
 214 following vertex-coloring of G . Use colors $1, 2, \dots, \ell$ to color each vertex in B ,
 215 color the vertices of A with color $\ell + 1$, the vertices of $V(K_p) \setminus B$ with color $\ell + 2$,
 216 and color the remaining vertices arbitrarily (e.g., all of them have color 1). Thus,
 217 pairs of vertices in $(A \cup V(K_p)) \times V(G)$ are rainbow vertex-connected. As for
 218 $x_1, x_2 \in W \setminus A$, let $y_1 \in N_{A \cup B}(x_1)$, $y_2 \in N_{K_p}(x_2)$. Then, there is a vertex-rainbow
 219 (x_1, x_2) -path containing y_1 and y_2 . So, $\text{rvc}(G) \leq \ell + 2 \leq r + 1 \leq \text{diam}(G) + r$.

220 The proof is complete. \blacksquare

221 Now let G be an $(S_{1,2,2}, N)$ -free graph, let $x, y \in V(G)$, and let $P : x =$
 222 $v_0, v_1, \dots, v_k = y$ ($k \geq 3$) be a shortest (x, y) -path in G . Let $z \in V(G) \setminus V(P)$. If
 223 $|N_P(z)| \geq 2$ and $\{v_i, v_j\} \subset N_P(z)$, then $|i - j| \leq 2$ and $|N_P(z)| \leq 3$, since P is a
 224 shortest path. Moreover, the following facts are easily observed.

- 225 • If $|N_P(z)| = 1$, then, since G is $S_{1,2,2}$ -free, z is adjacent to x, v_1, v_{k-1} or y .
- 226 • If $|N_P(z)| = 3$, then the vertices of $N_P(z)$ must be consecutive on P , since
 227 P is a shortest path.

228 This motivates the following notations:

- 229 • $A_i := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_i\}\}$ for $i = 0, 1, k - 1, k$;

- 230 • $L_i := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_{i-1}, v_{i+1}\}\}$ for $1 \leq i \leq k-1$;
 231 • $M_i := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_{i-1}, v_i\}\}$ for $1 \leq i \leq k$;
 232 • $N_i := \{z \in V(G) \setminus V(P) \mid N_P(z) = \{v_{i-1}, v_i, v_{i+1}\}\}$ for $1 \leq i \leq k-1$.
 233 We further set $S = V(P) \cup N_G(P)$ and $R = V(G) \setminus S$.

234 **Lemma 1.** Let G be an $(S_{1,2,2}, N)$ -free graph, let $x, y \in V(G)$ be such that
 235 $\text{dist}_G(x, y) \geq 4$ and let $P : x = v_0, v_1, \dots, v_k = y$, be a shortest (x, y) -path in G .
 236 Then

- 237 (i) $N_G(M_i) \subset S$, $i = 2, \dots, k-1$;
 238 (ii) $N_G(N_i) \subset S$, $i = 2, \dots, k-2$;
 239 (iii) $N_G(L_i) \subset S$, $i = 1, \dots, k-1$;
 240 (iv) $N_P(R) = \emptyset$;
 241 (v) $N_S(R) \subset A_0 \cup M_1 \cup N_1 \cup N_{k-1} \cup M_k \cup A_k$.

242 **Proof.** If $zv \in E(G)$ for some $z \in R$ and $v \in M_i$, $2 \leq i \leq k-1$, then we
 243 have $G[\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v, z\}] \simeq N$, a contradiction. Hence, (i) follows. To
 244 show (ii), we observe that if $zv \in E(G)$ for some $z \in R$ and $v \in N_i$, $2 \leq$
 245 $i \leq k-2$, then we have $G[\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v, z\}] \simeq S_{1,2,2}$, a contradic-
 246 tion. Similarly, for (iii), if $zv \in E(G)$ for some $z \in R$ and $v \in L_i$, $1 \leq$
 247 $i \leq k-1$, then, for $i = 1$ we have $G[\{v_1, v_2, v_3, v_4, v, z\}] \simeq S_{1,2,2}$, for $2 \leq$
 248 $i \leq k-2$ we have $G[\{z, v, v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}\}] \simeq S_{1,2,2}$, and for $i = k-1$,
 249 $G[\{v_{k-1}, v_{k-2}, v_{k-3}, v_{k-4}, v, z\}] \simeq S_{1,2,2}$, a contradiction. Part (iv) follows im-
 250 mediately from the definition of R , and by (i) through (iii), we have $N_S(R) \subset$
 251 $A_0 \cup A_1 \cup M_1 \cup N_1 \cup N_{k-1} \cup M_k \cup A_{k-1} \cup A_k$. But if $zv \in E(G)$ for some $z \in R$
 252 and $v \in A_1$, then $G[\{v_0, v_1, v_2, v_3, v, z\}] \simeq S_{1,2,2}$, a contradiction. Similarly, we
 253 have $N_{A_{k-1}}(R) = \emptyset$, implying (v).

254 The proof is complete. ■

255 **Theorem 8.** Let G be a connected $(S_{1,2,2}, N)$ -free graph. Then, $\text{rvc}(G) \leq$
 256 $\text{diam}(G) + 11$.

257 **Proof.** Let G be a connected $(S_{1,2,2}, N)$ -free graph. If $\text{diam}(G) \leq 2$, then
 258 $\text{rvc}(G) = \text{diam}(G) - 1$. Thus, for the rest of the proof we suppose that $\text{diam}(G) =$
 259 $d \geq 3$. Let $v_0, v_d \in V(G)$ be such that $\text{dist}_G(v_0, v_d) = d$, let $P : v_0 v_1 v_2 \dots v_d$ be a
 260 diameter path in G , and let A_i, L_i, M_i, N_i, S, R be defined as above.

261 We distinguish three cases according to the value of d .

262 **Case 1.** $d = 3$.

263 First, we partition $V(G)$ into four parts $P, N_G(P), N_G^2(P)$ and $N_G^3(P)$ accord-
 264 ing to the distance from P . Then, for the vertices in $N_G(P)$, we can partition
 265 them into three parts $X_1 = A_0 \cup M_1 \cup L_1 \cup N_1$, $X_2 = A_3 \cup M_3 \cup L_2 \cup N_2$ and
 266 $X_3 = A_1 \cup M_2 \cup A_2$. We must point out that $X_1 \cap X_2 = \emptyset$ and $N_R(X_3) = \emptyset$,

267 whose proof is similar to that of Lemma 1. Then, we denote Y_i the set of
 268 vertices in $N_G^2(P)$ such that for each $v \in Y_i$, $N_{N(P)}(v) \subset X_i, i = 1, 2$, and
 269 $Y_3 = N_G^2(P) \setminus (Y_1 \cup Y_2)$. With a similar reason as above, $N_{N_G^3(P)}(Y_3) = \emptyset$.
 270 So, analogously we can partition $N_G^3(P)$ into three parts Z_1, Z_2 and Z_3 . It
 271 should be noticed that $Z_1 = \emptyset$; otherwise there exists a vertex $z \in Z_1$ such
 272 that $\text{dist}_G(z, v_3) \geq 4$, a contradiction. Symmetrically, we have $Z_2 = \emptyset$.

273 Now, we define a vertex-coloring of G that uses at most 14 colors. Color
 274 the vertices of P with colors 0, 1, 2, 3 and color the vertices in $A_0, M_1, L_1, N_1, N_2,$
 275 L_2, M_3, A_3, Y_1 and Y_2 with colors 4, 5, \dots , 13, respectively. Then, color the re-
 276 maining vertices arbitrarily (e.g., all of them have color 0). We can show that
 277 this vertex-coloring can make G rainbow vertex-connected. We only need to
 278 verify that for a pair of vertices $x, y \in (Y_1 \times Y_1) \cup (Y_2 \times Y_2)$, there exists a
 279 vertex-rainbow path connecting them. Without loss of generality, we suppose
 280 $(x, y) \in Y_1 \times Y_1$. If $\text{dist}_G(x, y) \leq 2$, then there is nothing left to do. Next
 281 we consider $\text{dist}_G(x, y) \geq 3$. Let x' be an arbitrary neighbor of x in X_1 , and
 282 y' an arbitrary neighbor of y in X_1 . We claim that x' and y' cannot have the
 283 same color. Otherwise, we suppose that x' and y' are colored with the same
 284 color, i.e., they are in the same vertex-class of X_1 , and let $i = \max\{j : v_j \in$
 285 $N_P(x') \cap N_P(y')\}$. Then, we have $G[\{v_i, v_{i+1}, x', y', x, y\}] \simeq S_{1,2,2}$ if $x'y' \notin E(G)$,
 286 or $G[\{v_i, v_{i+1}, x', y', x, y\}] \simeq N$ if $x'y' \in E(G)$, respectively. So, the colors of x'
 287 and y' must be different. Then, the (x, y) -path $P_1 : xx'v_0y'y$ is vertex-rainbow.
 288 Hence, we have $\text{rvc}(G) \leq \text{diam}(G) + 11$.

289 **Case 2.** $d = 4$.

290 Similarly, with the partition and the vertex-coloring of Case 1, we can get
 291 that $\text{rvc}(G) \leq 15 = \text{diam}(G) + 11$.

292 **Case 3.** $d \geq 5$.

293 Set $B_c = (\cup_{i=2}^{d-2} N_i) \cup (\cup_{i=2}^{d-1} M_i) \cup (\cup_{i=1}^{d-1} L_i) \cup A_1 \cup A_{d-1} \cup \{v_1, v_2, \dots, v_{d-1}\}$,
 294 $X = A_0 \cup M_1 \cup N_1 \cup N_{d-1} \cup M_d \cup A_d$, $X_1 = A_0 \cup M_1 \cup N_1$, and $X_2 = N_{d-1} \cup M_d \cup A_d$.
 295 By virtue of Lemma 1, we have $N_G(B_c) \subset S$.

296 **Subcase 3.1.** B_c is a cut-set of G .

297 We claim that $S \cup N_G(S) = V(G)$. Suppose, to the contrary, that $z \in R$ is
 298 at distance 2 from S . Then, by Lemma 1 and the assumption of Case 1, as well
 299 as the symmetry, we can assume that $N_S^2(z) \subset X_1$. Let Q be a shortest (z, v_d) -
 300 path, let w be the first vertex of Q in B_c (it exists by the assumption of Subcase
 301 3.1), and let w^- be the predecessor of w on Q . By Lemma 1, $\text{dist}(w^-, P) = 1$,
 302 implying $w^- \in X_1$. Then, $\text{dist}_G(w^-, v_d) \geq d-1$; otherwise, the path $v_0w^-Qv_d$ is a
 303 (v_0, v_d) -path shorter than P . Since $\text{dist}_G(z, w^-) \geq 2$, we have $\text{dist}_G(z, v_d) \geq d+1$,
 304 contradicting $\text{diam}(G) = d$. Hence, we have $S \cup N_G(S) = V(G)$. Moreover, with
 305 a similar argument to that of Case 1, we have that for $x, y \in R$ with distance at
 306 least 3, their neighbors x' and y' cannot be in the same vertex-class of X .

307 Now we define a vertex-coloring of G that uses at most $d + 7$ colors. Color

308 the vertices of P with colors $0, 1, \dots, d$ and color the vertices in A_0, M_1, N_1, N_{d-1} ,
 309 M_d and A_d with colors $d+1, d+2, \dots, d+6$, respectively. Then, color the
 310 remaining vertices arbitrarily (e.g., all of them have color 0). We can show
 311 that this vertex-coloring can make G rainbow vertex-connected. For any pair of
 312 vertices in $S \times (S \cup R)$, we can easily find a vertex-rainbow path connecting them.
 313 For a pair $(x, y) \in R \times R$, if $\text{dist}_G(x, y) \leq 2$, then there is nothing left to do.
 314 Next we consider $\text{dist}_G(x, y) \geq 3$. From above, we know that their neighbors x'
 315 and y' in X are colored differently. So, the (x, y) -path containing x' and y' is
 316 vertex-rainbow.

317 Consequently, we have $\text{rvc}(G) \leq \text{diam}(G) + 7$.

318 **Subcase 3.2.** B_c is not a cut-set of G .

319 Set $H = G - B_c$. Let $P' : v_d v_{d+1} \dots v_{d+\ell-1} v_{d+\ell} = v_0$ be a shortest (v_d, v_0) -
 320 path in H . Since P is a diameter path, $\ell \geq d \geq 5$. If v_{d+1} is adjacent to v_{d-2} , then
 321 $G[\{v_d, v_{d+1}, v_{d-2}, v_{d-3}, v_{d+2}, v_{d+3}\}] \simeq S_{1,2,2}$, a contradiction. So, $v_{d+1} \in A_d \cup M_d$.
 322 Similarly, we have $v_{d+\ell-1} \in A_0 \cup M_1$.

323 Set $P^d : v_{d-1} v_d v_{d+1}$ if $v_{d-1} v_{d+1} \notin E(G)$, or $P^d : v_{d-1} v_{d+1}$ if $v_{d-1} v_{d+1} \in E(G)$,
 324 respectively. Similarly, set $P^0 : v_{d+\ell-1} v_0 v_1$ if $v_{d+\ell-1} v_1 \notin E(G)$, or $P^d : v_{d+\ell-1} v_1$
 325 if $v_{d+\ell-1} v_1 \in E(G)$, respectively. Finally, set $C : v_1 P v_{d-1} P^d v_{d+1} P' v_{d+\ell-1} P^0 v_1$.
 326 Then, C is a cycle of length at least $2d - 2$.

327 **Claim 1.** The cycle C is chordless.

328 **Proof.** This proof can be found in [5]. But for the sake of completeness, we
 329 provide the proof here. Suppose, to the contrary, that $v_i v_j \in E(G)$ is a chord
 330 in C . Since both P and P' are chordless, we can choose the notation such that
 331 $1 \leq i \leq d - 1$ and $d + 1 \leq j \leq d + \ell - 1$. Since $v_j \in V(P')$, we have $v_j \notin B_c$ by
 332 the definition of P' , implying $i = d - 1$ and $v_j \in M_d$, or, symmetrically, $i = 1$
 333 and $v_j \in M_1$. This implies that in the first case, $v_j = v_{d+1}$; in the second case,
 334 $v_j = v_{d+\ell-1}$; and in both cases, $v_i v_j \in E(C)$ by the definition of C . Thus, C is
 335 chordless. \square

336 **Claim 2.** $\ell \leq d + 2$.

337 **Proof.** Assume that $\ell \geq d + 3$, and let Q be a shortest (v_0, v_{d+2}) -path in G . Then,
 338 $|E(Q)| \leq d$ (since $\text{diam}(G) = d$). Since $\ell \geq d + 3$ and P' is shortest in $H = G - B_c$,
 339 we have $\text{dist}_H(v_0, v_{d+2}) \geq d + 1$. So, Q must contain a vertex from B_c . Let w be the
 340 last vertex of Q in B_c , and let w^- and w^+ be its predecessor and successor on Q ,
 341 respectively (they exist since $v_{d+2} \notin B_c$ by the definition of P'). By Lemma 1, w^+
 342 is at distance at most 1 from P . Since clearly $w^+ \notin \{v_0, v_d\}$, either $w^+ v_0 \in E(G)$
 343 or $w^+ v_d \in E(G)$. If $w^+ v_0 \in E(G)$, then $v_0 w^+ Q v_{d+2}$ is a (v_0, v_{d+2}) -path shorter
 344 than Q , a contradiction. Thus, $w^+ v_d \in E(G)$. Now, $w^+ \neq v_{d+2}$ since P' is
 345 chordless, implying $\text{dist}_G(v_0, w^+) \leq d - 1$. On the other hand, $\text{dist}_G(v_0, w^+) \geq$
 346 $d - 1$; otherwise, $v_0 Q w^+ v_d$ is a (v_0, v_d) -path of length at most $d - 1$, contradicting

347 the fact that P is a diameter path. Hence, $\text{dist}_G(v_0, w^+) = d - 1$, implying
 348 that $\text{dist}_G(v_0, w) = d - 2$ and $w^+v_{d+2} \in E(Q)$. Since $v_{d+2}, v_{d+3} \in R$, we have
 349 $G[\{v_{d+3}, v_{d+2}, v_d, w^+, w, w^-\}] \simeq S_{1,2,2}$, a contradiction. Hence, $\ell \leq d + 2$. \square

350 **Claim 3.** $C \cup N_G(C) = V(G)$, and every vertex in $V(G) \setminus V(C)$ has at least
 351 2 neighbors in C .

352 **Proof.** Suppose that a vertex $x \in V(G) \setminus V(C)$ at distance 1 from C has exactly
 353 one neighbor in C , and set $N_C(x) = \{y\}$. Let $z_1, z_2 \in N_C^2(x)$, and let $z'_1, z'_2 \in$
 354 $N_C^3(x)$. Then, we have $G[\{x, y, z_1, z_2, z'_1, z'_2\}] \simeq S_{1,2,2}$, a contradiction.

355 Secondly, suppose, to the contrary, that $z \in V(G)$ is at distance 2 from
 356 C , and y is a neighbor of z at distance 1 from C . Then, $\text{dist}_G(z, P) \geq 2$;
 357 otherwise, $y = v_0$ or v_d , without loss of generality, we assume $y = v_0$. Then,
 358 v_1 must be adjacent to $v_{d+\ell-1}$, and thus, $G[\{z, y, v_1, v_2, v_{d+\ell-1}, v_{d+\ell-2}\}] \simeq N$, a
 359 contradiction. Hence, $z \in R$. If $y \in R$, then y is not adjacent to any of v_1, v_2
 360 and v_3 . If $y \notin R$, then we have $y \in X$. Without loss of generality, we assume
 361 $y \in X_2$. Then, y is not adjacent to any of v_1, v_2 and v_3 . Moreover, from above
 362 we know that y has at least 2 neighbors in C . Let $x_1, x_2 \in N_C(y)$ be the vertices
 363 closest to v_1 and v_3 , respectively. Let x'_1 and x'_2 be their neighbors that are
 364 closer to v_1 and v_3 in C , respectively. Then, $G[\{y, z, x_1, x_2, x'_1, x'_2\}] \simeq S_{1,2,2}$ if
 365 $x_1x_2 \notin E(G)$, or $G[\{y, z, x_1, x_2, x'_1, x'_2\}] \simeq N$ if $x_1x_2 \in E(G)$, respectively. Thus,
 366 C is a dominating set of G . \square

367 By Claims 1 and 2, we know that C is a chordless cycle of length at most
 368 $d + \ell \leq 2d + 2$. Now, we define a vertex-coloring of G that uses at most $d + 1$ colors.
 369 Relabel $C = x_1x_2 \dots x_kx_{k+1} (= x_1)$, $8 \leq 2d - 2 \leq k \leq 2d + 2$. Then, we assign
 370 color i to the vertex x_i if $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ and assign color $i - \lfloor \frac{k}{2} \rfloor$ to x_i if $\lfloor \frac{k}{2} \rfloor < i \leq k$.
 371 We color the remaining vertices arbitrarily. We can show that this vertex-coloring
 372 can make G rainbow vertex-connected. From Theorem 1 and Claim 3, we know
 373 that under this vertex-coloring, pairs in $C \times V(G)$ are rainbow vertex-connected.
 374 For each vertex $z \in N_G(C)$, we may strengthen the result of Claim 3 that z has
 375 at least two neighbors colored differently in C . Otherwise, we suppose that z_1
 376 and z_2 are the only two neighbors of z having the same color in C . From the
 377 vertex-coloring, we know that $\text{dist}_C(z_1, z_2) = \lfloor \frac{k}{2} \rfloor \geq 4$. Then, we can easily find
 378 an induced $S_{1,2,2}$, a contradiction. So, for a pair $(x, y) \in N_G(C) \times N_G(C)$, we
 379 can find a vertex $x' \in N_C(x)$ and a vertex $y' \in N_C(y)$ such that x' and y' are
 380 colored differently. Since there exists a vertex-rainbow path P connecting x' and
 381 y' and the internal vertices of P are colored differently from x' and y' , the path
 382 $xx'Py'y$ is vertex-rainbow and connects x and y . Hence, $\text{rvc}(G) \leq d + 1$.

383 The proof of Theorem 8 is complete. \blacksquare

384 Combining Proposition 2 with Theorems 7 and 8, we have proved Theorem
 385 6.

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