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GENERALIZED RAINBOW CONNECTION OF GRAPHS AND THEIR COMPLEMENTS

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Abstract

Let G be an edge-colored connected graph. A path P in G is called ℓ-rainbow if each subpath of length at most $\ell + 1$ is rainbow. The graph G is called (k, ℓ)-rainbow connected if there is an edge-coloring such that every pair of distinct vertices of G are connected by k pairwise internally vertex-disjoint ℓ-rainbow paths in G. The minimum number of colors needed to make G (k, ℓ)-rainbow connected is called the (k, ℓ)-rainbow connection number of G and denoted by $rc_{k,\ell}(G)$. In this paper, we first focus on the $(1, 2)$-rainbow connection number of G depending on some constraints of $\overline{G}$. Then, we characterize the graphs of order n with $(1, 2)$-rainbow connection number $n-1$ or $n-2$. Using this result, we investigate the Nordhaus-Gaddum-Type problem of $(1,2)$-rainbow connection number and prove that $rc_{1,2}(G)+rc_{1,2}(\overline{G}) \leq n+2$ for connected graphs G and $\overline{G}$. The equality holds if and only if G or $\overline{G}$ is isomorphic to a double star.

Keywords: ℓ-rainbow path; (k,ℓ)-rainbow connected; (k,ℓ)-rainbow connection number.

2010 Mathematics Subject Classification: 04C15, 05C40.

1. Introduction

All graphs in this paper are finite, undirected, simple and connected. We follow the notation and terminology in the book [3].

When considering the transmission of information between agencies of the government, an immediate question is put forward as follows: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct? This question can be represented by a graph and studied by means of what is called rainbow colorings introduced by Chartrand et al. in [5]. An *edge-coloring* of a graph is a mapping from its edge set to the set of natural numbers (colors). A path in an edge-colored graph with no two edges sharing the same color is called a *rainbow path*. A graph G with an edge-coloring c is said to be *rainbow connected* if every pair of distinct vertices of G is connected by at least one rainbow path in G. The coloring c is called a *rainbow coloring* of the graph G. For a connected graph G, the minimum number of colors needed to make G rainbow connected is defined as the *rainbow connection number* of G and denoted by $rc(G)$. Many researchers have studied problems on rainbow connection. See [9, 12, 14] for example. For more details we refer to the survey paper [13] and the book [14].

The following question provides a relaxation of this concept: What is the minimum number of passwords or firewalls that allows one or more secure paths between every two agencies such that as we progress from one agency to another along such a path, we are required to change passwords at each step? Inspired by this, Borozan et al. in [2] and Andrews et al. in [1] introduced the concept of proper-path coloring of graphs. Let G be an edge-colored graph. A path P in G is called a *proper path* if no two adjacent edges of P are colored with the same color. An edge-colored graph G is *k-proper connected* if every pair of distinct vertices u, v of G are connected by k pairwise internally vertex-disjoint proper (u,v)-paths in G. For a connected graph G, the minimum number of colors needed to make G k-proper connected is called the *k-proper connection number* of G and denoted by $pc_k(G)$. Particularly for $k=1$, we write $pc_1(G)$, the proper connection number of G, as $pc(G)$ for simplicity. Recently, many results have been obtained on the proper connection number. For details, we refer to the dynamic survey [10].

Relaxing the notion of a rainbow path, the (k, ℓ)-proper-path coloring was defined in [11] as a generalization of rainbow coloring and proper-path coloring. The notion of ℓ-rainbow colorings was also independently defined and studied in [4, 6, 7]. A path P in G is called an *ℓ-rainbow path* if each subpath of length at most $\ell + 1$ is rainbow colored. The graph G is called *(k, ℓ)-rainbow connected* if there is an edge-coloring c such that every pair of distinct vertices of G are connected by k pairwise internally vertex-disjoint ℓ-rainbow paths in G. This coloring is called a *(k, ℓ)-rainbow-path coloring* of G. In addition, if t colors are used, then c is referred to as a *(k, ℓ)-rainbow-path t-coloring* of G. For a connected graph G, the minimum number of colors needed to make G (k, ℓ)-rainbow connected is called the *(k, ℓ)-rainbow connection number* of G and denoted by $rc_{k,\ell}(G)$. Particularly, for $k = 1$ and $\ell = 2$, there is an edge-coloring using $rc_{1,2}$ colors such that there exists a 2-rainbow path between each pair of vertices of the graph G. Furthermore, if we ensure that every path in G is a 2-rainbow path, then such an edge-coloring is called a *strong edge-coloring*. In addition, the strong chromatic index $\chi'_s(G)$, which was introduced by Fouquet and Jolivet [8], is the minimum number of colors needed in a strong edge-coloring of G. Immediately we get that $rc_{1,2}(G) \leq \chi'_s(G)$. And this inspires us to pay our attention to the $(1, 2)$-rainbow connection number of the connected graph G, i.e., $rc_{1,2}(G)$.

As an example of this concept, we consider the $(2, 3)$-rainbow connection number of the cycle C_{12}. Since $\ell = 3$, then each pair of edges with the same color must have at least 3 edges in between. Additionally, there are pairs of vertices at distance greater than 4, we see that $rc_{2,3}(C_{12}) \geq 4$. On the other hand, if we color the edges of C_{12} by alternating through the colors like $1, 2, 3, 4, 1, \ldots, 4$ in order around the cycle, then it is easy to see that this is a $(2, 3)$-rainbow connected coloring using 4 colors, so $rc_{2,3}(C_{12}) = 4$.

In this paper, we consider the (k, ℓ)-rainbow connection number of graphs and their complements. This paper is organized as follows. In Section 2, we list some useful results about the (k, ℓ)-rainbow connection number of a graph. In Section 3, we focus on $rc_{1,2}(G)$ depending on some constraints of $\overline{G}$. In Section 4, we first characterize the graphs of order n with $(1, 2)$-rainbow connection number $n-1$ or $n-2$. Using this result, we give the Nordhaus-Gaddum-Type result for the $(1, 2)$-rainbow connection number, i.e., $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq n + 2$ for connected graphs G and $\overline{G}$, and the equality holds if and only if G or $\overline{G}$ is isomorphic to a double star.

2. Preliminaries

In this section, we introduce some definitions and present several results which will be used later. Let G be a connected graph. We denote by n the number of its vertices and m the number of its edges. The *distance between two vertices u and v in G*, denoted by $d(u,v)$, is the length of a shortest path between them in G. The *eccentricity* of a vertex v is $ecc(v) := max_{x\in V(G)} d(v,x)$. The *radius* of G is $rad(G) := min_{x\in V(G)} ecc(x)$. We also write $\sigma_2'(G)$ as the largest sum of degrees of vertices x and y, where x and y are taken over all couples of adjacent vertices in G. Additionally, we set $[n] = \{1, 2, \cdots, n\}$ for any integer $n \geq 1$.

The following are some results that we will use in our proofs. The first is a simple observation that the addition of edges cannot increase the rainbow connection number.

Proposition 2.1 [11]**.** If G is a nontrivial connected graph and H is a connected spanning subgraph of G, $\ell \geq 1$ is an integer. Then $rc_{1,\ell}(G) \leq rc_{1,\ell}(H)$. Particularly, $rc_{1,\ell}(G) \leq rc_{1,\ell}(T)$ for every spanning tree T of G.

When we focus on trees, the following holds.

Theorem 2.2 [11]**.** If T is a nontrivial tree, then $rc_{1,2}(T) = \sigma_2'(T) - 1$.

For complete bipartite graphs, the situation is trickier.

Theorem 2.3 [11]**.** Let $\ell \geq 2$ be an integer and $m \leq n$. Then,

$$rc_{1,\ell}(K_{m,n}) = \begin{cases} n & if\ m = 1, \\ 2 & if\ m \geq 2\ and\ m \leq n \leq 2^m, \\ 3 & if\ \ell = 2,\ m \geq 2\ and\ n > 2^m \\ & or\ \ell \geq 3,\ m \geq 2\ and\ 2^m < n \leq 3^m, \\ 4 & if\ \ell \geq 3,\ m \geq 2\ and\ n > 3^m. \end{cases}$$

For a general 2-connected graph, we gave in [11] an upper bound for the $(1,2)$-rainbow connection number.

Theorem 2.4 [11]**.** If a graph G is 2-connected, then $rc_{1,2}(G) \leq 5$.

3. $(1,2)$-rainbow connection number for the complement of a graph

In this section, we investigate the $(1,2)$-rainbow connection number of G depending on some properties of its complement $\overline{G}$.

Theorem 3.1. If G is a graph with $diam(\overline{G}) \geq 4$, then $rc_{1,2}(G) \leq 3$.

Proof. We first claim that G must be connected. If not, $\overline{G}$ must contain a spanning complete bipartite graph which implies that $diam(\overline{G}) \leq 2$, a contradiction. Choose a vertex x with $ecc_{\overline{G}}(x) = diam(\overline{G})$. Let $N_i(x) = \{v : dist_{\overline{G}}(x, v) = i\}$ for $0 \leq i \leq 3$ and $N_4(x) = \{v : dist_{\overline{G}}(x, v) \geq 4\}$. Obviously $N_0(x) = \{x\}$. We write N_i (for $0 \leq i \leq 4$) instead of $N_i(x)$ and n_i instead of $|N_i|$ for convenience. It can be deduced that all edges are present in G of the form uv where $u \in N_1$ and $v \in N_3 \bigcup N_4$ or $u \in N_2$ and $v \in N_4$ (see Figure 1).

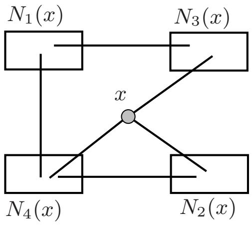


Figure 1. The graph G for the proof of Theorem 3.1.

We denote by $N_{i,j}(0 \leq i \neq j \leq 4)$ the edge set between N_i and N_j in G. We distinguish four cases and give each of the cases a $(1, 2)$-rainbow-path 3-coloring, respectively. Again we use $f(e)(e \in E(\overline{G}))$ to represent the color assigned to e.

Case 1. If $n_4 > 1$. We give all edges of $N_{1,3}$ the color 3, edges of $N_{0,3}$ the color 3, edges of $N_{0,4}$ the color 2, edges of $N_{0,2}$ the color 3, edges of $N_{2,4}$ the color 1. Additionally, color the edges of $N_{1,4}$ such that for $v \in N_1$, $\{f(vs) : s \in N_4\} = \{1, 2\}$. Then for any $u, v \in N_1(if\ n_1 > 1)$, there must exist $s_1, s_2 \in N_4$ (possibly with $s_1 = s_2$) such that $f(us_1) = 1$ and $f(vs_2) = 2$. Then one of us_1v or us_1xss_2v, where $s \in N_2$, is a 2-rainbow (u, v)-path. Other situations can be checked similarly.

Case 2. If $n_4 = 1$, $n_3 > 1$ and $n_1 = 1$. Then we give all edges of $N_{1,3}$ the color 1, the edge of $N_{1,4}$ the color 3, edges of $N_{0,3}$ the color 1, edges of $N_{0,4}$ the color 2, edges of $N_{0,2}$ the color 1 and edges of $N_{2,4}$ the color 3. It is easy to verify this is indeed a $(1, 2)$-rainbow-path 3-coloring of G.

Case 3. If $n_4 = 1$, $n_3 > 1$ and $n_1 > 1$. Let G' be the complete bipartite graph $G' = G[N_1 \cup N_3]$. By Theorem 2.3, we can use at most three colors to make G' $(1, 2)$-rainbow connected. Then we give all edges of $N_{1,4}$ the color 1, edges of $N_{0,3}$ the color 2, the edge of $N_{0,4}$ the color 3, edges of $N_{0,2}$ the color 1 and edges of $N_{2,4}$ the color 2. One can easily check this is a $(1, 2)$-rainbow-path 3-coloring of G and we omit the details here.

Case 4. If $n_4 = 1$ and $n_3 = 1$. Then we give all edges of $N_{1,3}$ the color 1, edges of $N_{1,4}$ the color 1, the edge of $N_{0,3}$ the color 2, the edge of $N_{0,4}$ the color 3, edges of $N_{0,2}$ the color 2 and edges of $N_{2,4}$ the color 1. We can again verify the correctness easily.

Thus, the proof is completed. □

Theorem 3.2. For a graph G, if $\overline{G}$ is triangle-free and $diam(\overline{G}) = 3$, then $rc_{1,2}(G) \leq 3$.

Proof. As in the proof of Theorem 3.1, it is easy to show that G is connected. Choose a vertex x such that $ecc_{\overline{G}}(x) = diam(\overline{G}) = 3$. In addition, N_i, n_i and $N_{i,j}$ for $0 \leq i \neq j \leq 3$ are defined as in the previous theorem. Again it can be deduced that there exist all edges of the form uv where $u \in N_0$ and $v \in N_2 \cup N_3$ or where $u \in N_1$ and $v \in N_3$. Since $\overline{G}$ is triangle-free and x has all edges to N_1 in $\overline{G}$, we know that N_1 is a clique in G. We give a $(1, 2)$-rainbow-path 3-coloring for G as follows.

We assign to the edges of $N_{0,2}$ the color 3, edges of $N_{0,3}$ the color 1, edges of $N_{1,3}$ the color 2, any edges of $N_{1,2}$ the color 3, any edges of $N_{2,3}$ the color 2 and the edges of the induced subgraph $G[N_1]$ the color 3.

It is obvious that for any $u \in N_i$ and $v \in N_j (i \neq j)$, there exists a 2-rainbow path between them. Then it suffices to show that for any $u, v \in N_2$ or N_3, there is a 2-rainbow path connecting them in G. First suppose $u, v \in N_2$ and there is no edge between them in G. Since $\overline{G}$ is triangle-free, there exists a vertex $w \in N_1$ such that $wv \in G$, then $uxtwv$ is a 2-rainbow path between u and v, where $t \in N_3$. The situation for any vertices $u, v \in N_3$ can be dealt with similarly. Thus $rc_{1,2}(G) \leq 3$. □

Theorem 3.3. Let G be a connected graph. If $\overline{G}$ is triangle free and $diam(\overline{G}) = 2$, then $rc_{1,2}(G) \leq 3$.

Proof. First we choose a vertex x with $ecc_{\overline{G}}(x) = diam(\overline{G}) = 2$. In addition, N_i, n_i and $N_{i,j}$ are defined as above. Clearly, all edges of the form xv for $v \in N_2$ are present in G. Again N_1 is a clique in G since all edges of the form xu are in $\overline{G}$ for $u \in N_1$ and $\overline{G}$ is triangle free.

Suppose there exists a vertex $v_0 \in N_2$ such that no edge $vw(w \in N_1)$ exists in G. Then v_0 is adjacent to every vertex of N_1 in $\overline{G}$. Thus, since every vertex of N_2 has at least one edge to N_1 in $\overline{G}$, the vertex v_0 must be adjacent to every other vertex of N_2 in G since otherwise a triangle will appear in $\overline{G}$. Next we give an edge coloring f for G. We set $f(xv_0) = 3$, $f(xw) = 2$ and $f(v_0w) = 1$ $(w \in N_2, w \neq v_0)$. And we give any edges of $N_{1,2}$ the color 2, the edges of the induced subgraph $G[N_1]$ the color 3. We only need to consider the 2-rainbow path for $w_1, w_2 \in N_2$ and $w_1v_0xw_2$ clearly suffices.

Next suppose there exists no such vertex v_0. Since G and $\overline{G}$ connected, we know that $n_1 \geq 2$. We denote by $E_G(v)$ (for $v \in N_2$) the set of edges between v and vertices of N_1 in G and set $e_G(v) = |E_G(v)|$. Also $e_{\overline{G}}(v)$ (for $v \in N_2$) is defined similarly. Again we distinguish two cases to analyze.

If $|N_1| \geq 3$, for each $u \in N_2$ with $e_G(u) = 1$, we give this edge the color 1. And for $u \in N_2$ with $e_G(u) \geq 2$, we arbitrarily color these edges but confirm that $\{f(e) : e \in E_G(u)\} = \{1, 2\}$. Then we set $f(xu) = 2$ $(u \in N_2)$ and give the edges of the induced subgraph $G[N_1]$ the color 3. The rest edges are colored arbitrarily with colors from [3]. Again we only need to consider the 2-rainbow path between the two non-adjacent vertices $v, w \in N_2$. Since $|N_1| \geq 3$ and v and w are non-adjacent in G, so $e_{\overline{G}}(v) + e_{\overline{G}}(w) \leq |N_1|$. Thus $e_G(v) + e_G(w) \geq |N_1| \geq 3$ which implies that one of the vertices v, w, say v, must have $e_G(v) \geq 2$. So there exists one vertex $s \in N_1$ or two vertices $s, t \in N_1$ such that vsw or $vstw$ is a 2-rainbow (v, w)-path in G.

If $|N_1| = 2$ and $N_1 = \{s, t\}$. Then each vertex $u \in N_2$ is adjacent to only one vertex of N_1 in G, either s or t since otherwise $diam(\overline{G}) \geq 3$. We denote by V_1 the set of vertices of N_2 adjacent to s in G, that is, the set adjacent to t in $\overline{G}$. And we write V_2 for the rest of the vertices of N_2. It is easy to see that V_1 and V_2 both induce cliques in G. We then set $f(xu)$ $(u \in V_1) = 1$, $f(us)$ $(u \in V_1) = 2$, $f(xu)$ $(u \in V_2) = 2$, $f(ut)$ $(u \in V_2) = 1$, $f(st) = 3$ and color any remaining edges with color 1. It is easy to check that this is a $(1, 2)$-rainbow-path 3-coloring of G. Thus the proof is completed. □

4. Nordhaus-Gaddum-Type theorem for $(1, 2)$-rainbow connection number

In this section, we first characterize the graphs on n vertices with $(1, 2)$-rainbow connection number $n - 1$ or $n - 2$, which is crucial to investigate the Nordhaus-Gaddum-Type result for the $(1, 2)$-rainbow connection number of the graph G. We use C_n, S_n to denote the cycle and the star graph on n vertices, respectively. Denote by $T(n_1, n_2)$ the double star in which the degrees of its (adjacent) center vertices are $n_1 + 1$ and $n_2 + 1$ respectively. Additionally, we write $T^1(n_1, n_2)$ as the graph obtained by replacing one pendent edge with P_3 in the double star $T(n_1, n_2)$ and denote the new pendent vertex by u_0 (see Figure 2). Also define graphs $G_1, \ldots, G_8$ as in Figure 2.

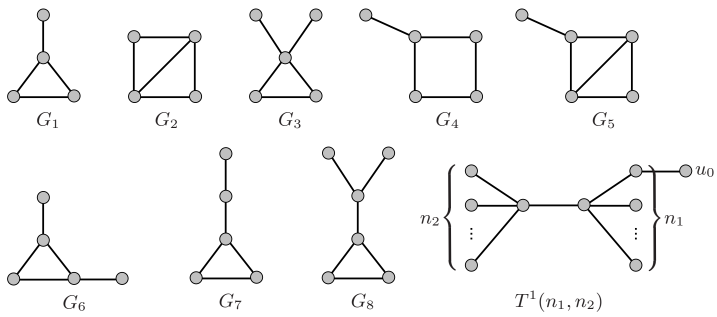


Figure 2. Graphs G_i $(1 \leq i \leq 8)$ and $T^1(n_1, n_2)$ in $\mathcal{G}_2$.

Theorem 4.1. Let G be a nontrivial connected graph on $n \geq 2$ vertices. Then

(*i*) $rc_{1,2}(G) = n - 1$ if and only if $G \in \mathcal{G}_1 = \{S_n \ (n \geq 2),\ T(n_1, n_2)\ (n_1, n_2 \geq 1)\}$;

(*ii*) $rc_{1,2}(G) = n - 2$ if and only if $G \in \mathcal{G}_2 = \{C_3,\ C_4,\ C_5,\ G_1,\ G_2,\ G_3,\ G_4,\ G_5,\ G_6, G_7,\ G_8,\ T^1(n_1, n_2)\}$.

Proof. Let G be a connected graph of order $n \geq 2$ and T be a spanning tree of G. Proposition 2.1 shows that $rc_{1,2}(G) \leq rc_{1,2}(T)$. Now we give proofs for (*i*) and (*ii*) separately.

Proof of (*i*): For any graph $G \in \mathcal{G}_1$, we can easily check that $rc_{1,2}(G) = n - 1$. So it remains to verify the converse. Since $rc_{1,2}(G) = n - 1$, we see that $n - 1 = rc_{1,2}(G) \leq rc_{1,2}(T) \leq n - 1$, i.e., $rc_{1,2}(T) = n - 1$. Thus, by Theorem 2.2, we know that any spanning tree T of G must be a star or a double star, i.e., $T \in \mathcal{G}_1$.Without loss of generality, we can assume that $n_2 \geq n_1$.

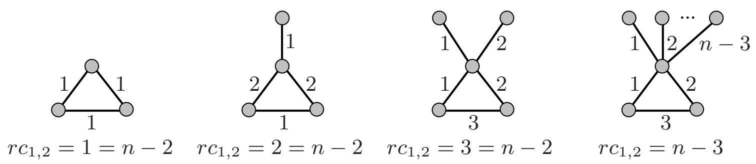


Figure 3. Graphs obtained by adding an edge to S_n $(n \geq 2)$.

If G is a tree, then $G \in \mathcal{G}_1$. Now we suppose that G is not a tree. Then since $T \in \mathcal{G}_1$, G can be constructed from S_n $(n \geq 2)$ or $T(n_1, n_2)$ $(n_1, n_2 \geq 1)$ by adding edges. Adding an edge to S_n $(n \geq 2)$, we will obtain one of the graphs depicted in Figure 3. However, all the graphs in Figure 3 have $(1, 2)$-rainbow connection number no more than $n - 2$, which implies that any spanning tree T of G cannot be a star. Next, we will consider the graphs obtained by adding edges to $T(n_1, n_2)$ $(n_1, n_2 \geq 1)$.

If $n_1 = n_2 = 1$, then $T(1,1) = P_4$. If an edge is added, then we will obtain either the cycle C_4 or the graph G_1 depicted in Figure 2. Obviously, both C_4 and G_1 have $(1,2)$-rainbow connection number $2 = n-2 < n-1$. For the cases $n_1 = 1$, $n_2 = 2$ and $n_1 = n_2 = 2$, one of the graphs in Figure 4 or 5 will be obtained by adding an edge to $T(1,2)$ or $T(2,2)$ respectively. The $(1,2)$-rainbow-path colorings given in Figures 4 and 5 show that all these graphs have $(1,2)$-rainbow connection number no more than $n-2$.

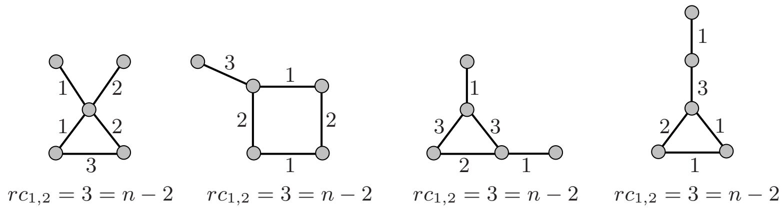


Figure 4. Graphs obtained by adding an edge to $T(1,2)$.

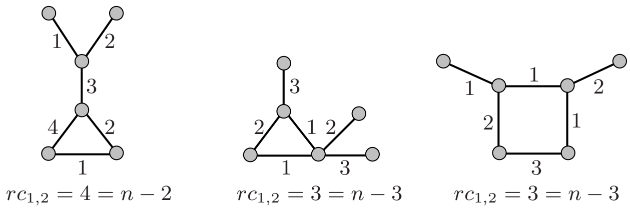


Figure 5. Graphs obtained by adding an edge to $T(2,2)$.

For all the other situations, i.e., $n_1 = 1$, $n_2 \geq 3$ or $n_1 = 2$, $n_2 \geq 3$ or $n_1 \geq 3$, $n_2 \geq 3$, Figure 6, Figure 7 and Figure 8 give all the graphs obtained by adding an edge to $T(1, n_2 \geq 3)$, $T(2, n_2 \geq 3)$ and $T(n_1 \geq 3, n_2 \geq 3)$, respectively. We give $(1,2)$-rainbow-path colorings for these graphs showed in Figure 6, Figure 7 and Figure 8. One can easily check that all these graphs have $(1,2)$-rainbow connection number no more than $n-2$.

From the discussions all above, we come to a conclusion that if $rc_{1,2}(G) = n-1$, then $G \in \mathcal{G}_1 = \{S_n\ (n \geq 2),\ T(n_1,n_2)(n_1,n_2 \geq 1)\}$.

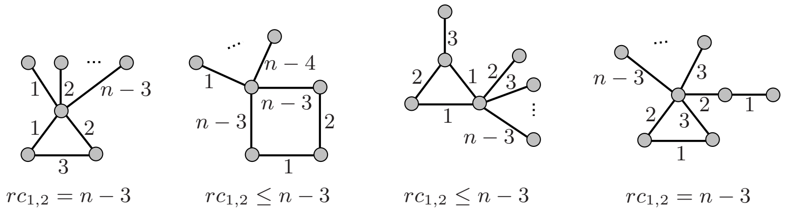


Figure 6. Graphs obtained by adding an edge to $T(1, n_2 \ge 3)$.

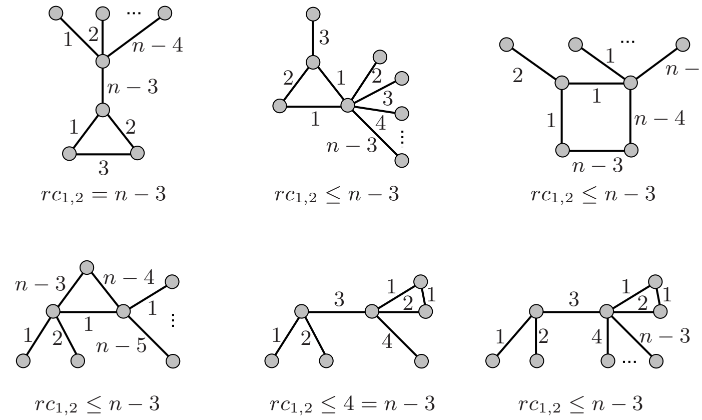


Figure 7. Graphs obtained by adding an edge to $T(2, n_2 \ge 3)$.

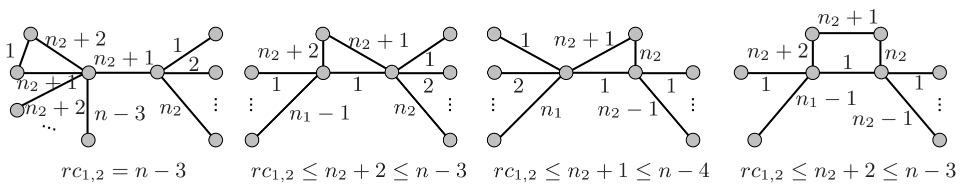


Figure 8. Graphs obtained by adding an edge to $T(n_1 \ge 3, n_2 \ge 3)$.

Proof of (*ii*): One can easily check that $rc_{1,2}(G) = n-2$ for any graph $G \in \mathcal{G}_2$. Hence, it remains to show the converse. Since $rc_{1,2}(G) = n-2$, then $n-2 \le rc_{1,2}(T) \le n-1$. Thus, Theorem 2.2 implies that any spanning tree T of G must be an element of the set $\{S_n\ (n \ge 2),\ T(n_1, n_2)\ (n_1, n_2 \ge 1),\ T^1(n_1, n_2)\ (n_1, n_2 \ge 1)\}$.

If G is a tree, then $G \cong T^1(n_1, n_2)\ (n_1, n_2 \ge 1) \subseteq \mathcal{G}_2$. Next we suppose that G is not a tree. Then G can be constructed from $S_n\ (n \ge 2)$,

$T(n_1, n_2)$ $(n_1, n_2 \geq 1)$ or $T^1(n_1, n_2)$ $(n_1, n_2 \geq 1)$ by adding edges. In the proof of (i), we listed eight graphs with $(1, 2)$-rainbow connection number $n - 2$, which are C_3, C_4, G_1, G_3, G_4, G_6, G_7 and G_8, respectively. Furthermore, all graphs obtained by adding an edge to S_n $(n \geq 2)$ or $T(n_1, n_2)$ $(n_1, n_2 \geq 1)$ except these eight ones have $(1, 2)$-rainbow connection number no more than $n - 3$. Therefore, the graph G can be constructed from C_3, C_4, G_1, G_3, G_4, G_6, G_7, G_8 or $T^1(n_1, n_2)$ $(n_1, n_2 \geq 1)$ by adding edges.

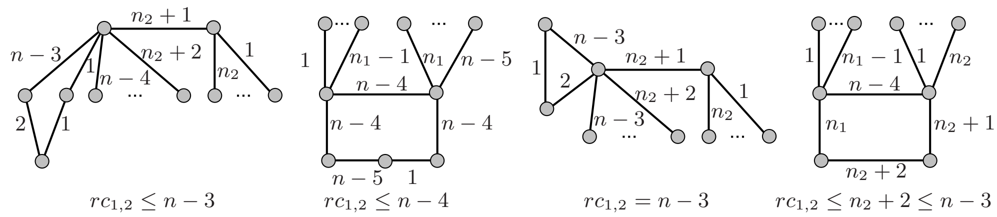


Figure 9. Graphs obtained by adding an edge to $T^1(n_1 \geq 2, n_2 \geq 2)$.

Considering graphs constructed from C_3, C_4, G_1, G_3, G_4, G_6, G_7 or G_8 by adding edges, we find only another two graphs G_2, G_5 with $rc_{1,2}(G_2) = 2 = |V(G_2)| - 2$ and $rc_{1,2}(G_5) = 3 = |V(G_5)| - 2$. All others have $(1, 2)$-rainbow connection number no more than $n - 3$. Now we focus on the graphs obtained by adding an edge to $T^1(n_1, n_2)$ $(n_1, n_2 \geq 1)$. For the cases $n_1 = n_2 = 1$, $n_1 = 1$, $n_2 \geq 2$ and $n_1 \geq 2$, $n_2 = 1$, we find another graph C_5 such that $rc_{1,2}(C_5) = n-2$ with similar analysis as in the proof of (i). Denote by e the new edge added to $T(n_1, n_2)$ $(n_1, n_2 \geq 1)$ or $T^1(n_1, n_2)$ $(n_1, n_2 \geq 1)$ and $T(n_1, n_2)+e$, $T^1(n_1, n_2) + e$ the newly obtained graphs. For the case $n_1 \geq 2$, $n_2 \geq 2$, we consider cases depending on whether the pendent vertex u_0 in $T^1(n_1, n_2)$ is an end vertex of e or not. It is obvious that if $u_0 \notin e$, then $T^1(n_1, n_2) + e \setminus u_0 \cong T(n_1, n_2)+e$. The proof of (i) suggests that we only need to consider the case when $T^1(n_1, n_2)+e \setminus u_0 \cong G_8$. It is easy to check that $rc_{1,2}(T^1(n_1, n_2)+e) = n-3 < n-2$ for this case. If $u_0 \in e$, then one of the graphs in Figure 9 will be obtained by adding an edge to $T^1(n_1, n_2)$. However, all these graphs have $(1, 2)$-rainbow connection number no more than $n - 3$ (as colored in the figure). Thus, we complete the proof of (ii). ∎

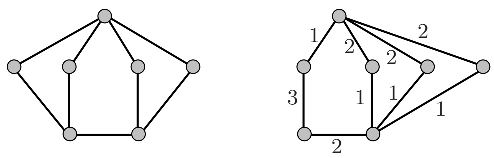
Theorem 4.2. Let G and $\overline{G}$ be connected graphs on n vertices. Then $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq n + 2$ and the equality holds if and only if G or $\overline{G}$ is isomorphic to a double star, i.e., $G \cong T(n_1, n_2)$ $(n_1, n_2 \geq 1)$ or $\overline{G} \cong T(n_1, n_2)$ $(n_1, n_2 \geq 1)$.

Proof. Since both G and $\overline{G}$ are connected, we have $n \geq 4$ and $\Delta(G)$, $\Delta(\overline{G}) \leq n - 2$. Let G be the double star with center vertices u, v and $N_G(u) \setminus v = A$, $N_G(v) \setminus u = B$. So, $\overline{G}[A \cup B]$ is a clique and $N_{\overline{G}}(u) = B$, $N_{\overline{G}}(v) = A$.

Certainly all edges of G must have distinct colors so we consider colorings of $\overline{G}$. Color all edges incident to v with 1, all edges incident to u with 2 and edges in $\overline{G}[A \cup B]$ with 3. This coloring shows that $rc_{1,2}(\overline{G}) \leq 3$. Since u and v are at distance 3 in $\overline{G}$, we get that $rc_{1,2}(\overline{G}) = 3$ and so $rc_{1,2}(G)+rc_{1,2}(\overline{G}) = n+2$. Now, we must show that $rc_{1,2}(G) + rc_{1,2}(\overline{G}) < n + 2$ for all other connected graphs G and $\overline{G}$. One can easily check that this is true for $n = 4,\ 5$. So we consider $n \geq 6$ in the following.

If G or $\overline{G}$ has $(1, 2)$-rainbow connection number $n - 1$ or $n - 2$, i.e., $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \setminus T(n_1, n_2)$ $(n_1, n_2 \geq 1)$ or $\overline{G} \in \mathcal{G}_1 \cup \mathcal{G}_2 \setminus T(n_1, n_2)$ $(n_1, n_2 \geq 1)$, then $rc_{1,2}(G) + rc_{1,2}(\overline{G}) < n + 2$ by simple examination. Hence, we can assume that $2 \leq rc_{1,2}(G) \leq n - 3$ and $2 \leq rc_{1,2}(\overline{G}) \leq n - 3$.

Suppose first that both G and $\overline{G}$ are 2-connected. For $n = 6$, it is easy to check that $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq 3 + 3 < 8 = n + 2$. And for $n \geq 9$, Theorem 2.4 implies that $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq 5 + 5 = 10 < 11 \leq n + 2$. Then what remains are the cases $n = 7$ and $n = 8$. For convenience, we denote the circumference of G by $c(G)$. We first suppose $n = 7$. Obviously $4 \leq c(G) \leq 7$. If $c(G) = 7$, then C_7 is a spanning subgraph of G and $rc_{1,2}(G) \leq rc_{1,2}(C_7) = 3$. If $c(G) = 6$, then G has a traceable spanning subgraph which is composed of C_6 by adding an open ear of length two. Thus, $rc_{1,2}(G) \leq 3$. If $c(G) = 5$, then G contains H_1^7 or H_2^7 (see Figure 10) as a spanning subgraph. Since H_1^7 is traceable and $rc_{1,2}(H_2^7) \leq 3$, then $rc_{1,2}(G) \leq 3$. For the case $c(G) = 4$, G contains $K_{2,5}$ as its spanning subgraph, which contradicts the assumption that $\overline{G}$ is connected. Therefore, all 2-connected graphs of order $n = 7$ with connected complementary graphs has $(1, 2)$-rainbow connection number no more than 3. Hence, $rc_{1,2}(G)+rc_{1,2}(\overline{G}) \leq 3+3 < 9 = n+2$. With similar analysis as for the situation $n = 7$, we can also draw the conclusion that $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq 3 + 3 < 10 = n + 2$ for $n = 8$.



H_1^7 H_2^7

Figure 10. Graphs for the proof of Theorem 4.2.

Now we consider the case where at least one of G and $\overline{G}$ has at least one cut vertex. Without loss of generality, suppose that G has at least one cut vertex. We distinguish the following two cases.

Case 1: G has a cut vertex u such that $G - u$ has at least three components.

Let $G_1,\ G_2, \cdots, G_k$ $(k \geq 3)$ be the components of $G - u$, and let n_i be the number of vertices of G_i for $i = 1, 2, \ldots, k$ with $n_1 \leq n_2 \leq \cdots \leq n_k$. Since

$\Delta(G) \leq n-2$, then $n_k \geq 2$. The complementary graph $\overline{G} \setminus u$ contains $K_{n_k,n-n_k-1}$ as a spanning subgraph and both $n_k \geq 2$ and $n - n_k - 1 \geq 2$. By Theorem 2.3, there exists a $(1,2)$-rainbow-path 3-coloring of $K_{n_k,n-n_k-1}$ using elements in [3]. Then, if we color the edges incident to u in $\overline{G}$ with color 4, then we obtain a $(1,2)$-rainbow-path 4-coloring of $\overline{G}$. Therefore, $rc_{1,2}(G)+rc_{1,2}(\overline{G}) \leq (n-3)+4 = n+1 < n+2$.

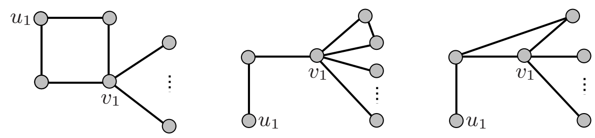
Case 2: Each cut vertex u of G satisfies that $G-u$ has only two components.

Let $G_1,\ G_2$ be the two components of $G-u$, and let n_i be the number of vertices of G_i for $i = 1,2$ with $n_1 \leq n_2$. Since $n \geq 6$, then $n_2 \geq 2$.

Subcase 2.1: $n_1 \geq 2$. The complementary graph $\overline{G} \setminus u$ contains K_{n_1,n_2} as a spanning subgraph. By Theorem 2.3, there is a coloring of K_{n_1,n_2} with colors in [3], and we color the edges incident to u in $\overline{G}$ with color 4. This gives a $(1,2)$-rainbow-path 4-coloring of $\overline{G}$. As a result, $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq n-3+4 = n+1 < n+2$ as desired.

Subcase 2.2: $n_1 = 1$, i.e., each cut vertex of G is incident with a pendent edge.

Since $n \geq 6$, then $n_2 \geq 4$. Let $\{u_1,\ u_2, \ldots, u_\ell\}$ be the set of all cut vertices of G, and let $u_1v_1,\ u_2v_2, \ldots, u_\ell v_\ell$ be the pendent edges incident to these cut vertices in G. Set $H = G \setminus \{v_1,\ v_2, \ldots, v_\ell\}$, so H is 2-connected. By Theorem 2.4, we know that $rc_{1,2}(H) \leq 5$.



G^1 G^2 G^3

Figure 11. Graphs for the proof of Theorem 4.2.

If $\ell \geq 2$, then $\overline{G} \setminus \{u_1, u_2\}$ contains $K_{2,n-4}$ as a spanning subgraph. By Theorem 2.3, there is a coloring of $K_{2,n-4}$ using colors from [3], and we color the edges incident to u_1 or u_2 in $\overline{G}$ with color 4. One can easily check this is a $(1,2)$-rainbow-path 4-coloring of $\overline{G}$. Thus, $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq (n-3)+4 = n+1 < n+2$.

Thus, we may assume $\ell = 1$, so $rc_{1,2}(G) \leq rc_{1,2}(H) + 1 \leq 6$. Since $\overline{G}$ is connected, then $|N_{\overline{G}}(u_1)| \geq 1$ and $\overline{G}$ contains G^1, G^2 or G^3 (see Figure 11) as a spanning subgraph. We first suppose that G^1 is a spanning subgraph of $\overline{G}$. Let $H_1, \ldots, H_5$ be as in Figure 12. If $\overline{G} \cong H_1$, then it is easy to verify that $rc_{1,2}(G) + rc_{1,2}(\overline{G}) = 3+3 = 6 < 8 = n+2$ for $n = 6$ and $rc_{1,2}(G) + rc_{1,2}(\overline{G}) = 4+3 = 7 < 9 = n+2$ for $n = 7$. If $\overline{G} \cong H_1$ and $n \geq 8$, the coloring depicted in Figure 12 shows that $rc_{1,2}(\overline{G}) \leq n-4$. In addition, if we color u_1v_1 with color 1, other edges incident to u_1 with color 2 and all other edges color 3 in G, then we get a $(1,2)$-rainbow-path 3-coloring of G. Consequently, $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq (n-4)+3 = n-1 < n+2$. Next we consider the situation $H_1 \subsetneqq \overline{G}$. Adding an edge to G^1, we arrive at some graph in $\{H_2,\ H_3,\ H_4,\ H_5\}$ depicted in Figure 12. If $\overline{G} \cong H_5$, then $rc_{1,2}(\overline{G}) \leq n-4$ by the coloring in Figure 12. In order to color G, we color u_1v_1 with color 1 and other edges incident to u_1 with color 2. Additionally, we color edges incident to x (y is the same) with colors $1,\ 3$ such that both 1 and 3 appear and all other edges with color 2 in G. Thus, we get a $(1,2)$-rainbow-path 3-coloring of G and so $rc_{1,2}(G) + rc_{1,2}(\overline{G}) \leq 3 + (n-4) = n-1 < n+2$. If $\overline{G}$ is not isomorphic to H_5, then $\overline{G}$ has $H_2,\ H_3$ or H_4 as its spanning subgraph. As is depicted in Figure 12, $rc_{1,2}(H_i) \leq n-5$ $(2 \leq i \leq 4)$ for $n \geq 9$. Therefore, $rc_{1,2}(G)+rc_{1,2}(\overline{G}) \leq 6+(n-5) = n+1 < n+2$ for $n \geq 9$. For the situation $6 \leq n \leq 8$, we can verify the result depending on the circumference of $H = G \setminus u_1$ similarly as above. Hence, if G^1 is a spanning subgraph of G, then $rc_{1,2}(G) + rc_{1,2}(\overline{G}) < n+2$. By the same method, we can draw the same conclusion for G^2 or G^3 as a spanning subgraph of G. Therefore, we complete the proof.

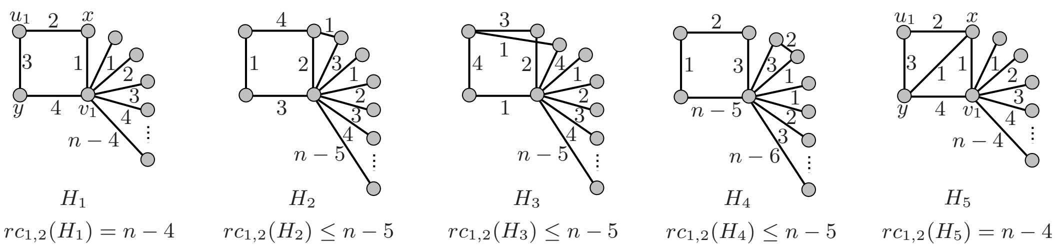


Figure 12. Graphs for the proof of Theorem 4.2.

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