Borderenergetic graphs with small maximum or large minimum degrees

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Abstract

The energy of a graph G is defined as the sum of the absolute values of all eigenvalues of the adjacency matrix of G. A graph is said to be borderenergetic if its energy is equal to that of the complete graph of the same order. In the first part of the paper, we study the existence of borderenergetic chemical graphs (a graph is chemical if it has a maximum degree at most 4). We show that there is no borderenergetic graph with maximum degree at most 3. We then provide five necessary conditions for borderenergetic graphs with maximum degree 4, and as a result, we show that there is no borderenergetic graph with maximum degree 4 and order $n \geq 22$. In the second part, we consider a problem contrary to the first part, i.e., borderenergetic graphs with large minimum degrees. We show that there is no borderenergetic graph of order n with minimum degree $n-2$. We then construct two families of borderenergetic graphs with minimum degree $n-3$ and $n-4$, respectively, the former is for all integers $n \geq 7$ while the latter is for all even numbers $n \geq 8$.

Keywords: borderenergetic graph; chemical graph; minimum degree; maximum degree.

1 Introduction

In this paper, all graphs are simple, undirected and finite. For notation and terminology we follow the book [2]. The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of the adjacency matrix of a graph G are said to be the *eigenvalues of the graph* G. These eigenvalues together with their multiplicities form the *spectrum of the graph* G, which is abbreviated as $Sp(G)$ in the following. We refer the book [3] for further details.

The *energy* of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of G. This concept has aroused wide attention owing to its remarkable practical application value in chemistry. It is also closely related to many aspects in graph theory and has been deeply studied. Tremendous results have been obtained and displayed in books [9] and [14], together with the paper [7].

It is well known that the complete graph of order n has energy $2n-2$. A graph G of order n is said to be *borderenergetic* if $\mathcal{E}(G) = 2n-2 = \mathcal{E}(K_n)$. A borderenergetic graph is called noncomplete if it is not a complete graph. This concept was introduced in [8] for the first time. Plenty of basic properties on the borderenergetic graphs have been derived in recent papers; see [5, 8, 11]. In [8], all noncomplete borderenergetic graphs of orders 7, 8 and 9 were presented. Furthermore, they also put forward effective tools to build borderenergetic graphs. In addition, we in [15] and Shao and Deng in [20] searched out all the borderenergetic graphs on 10 and 11 vertices by computers. Nikiforov [17] in 2007 showed that for almost all graphs,

$$\mathcal{E} = \Big(\frac{4}{3\pi} + o(1)\Big)n^{\frac{3}{2}}.$$

Thus the ratio of borderenergetic graphs is rather small according to the above equation.

Many results on graph energy are closely related to their maximum or minimum degrees. For instance, Nikiforov [18] obtained the following result: Let G be a graph of order n with at least n edges and no isolated vertices. If G is C_4-free and $\Delta(G) \leq 3$, then $\mathcal{E}(G) > n$. In [13], Li and Ma proved that there are exactly 4 connected graphs with maximum degree $\Delta \leq 3$ whose energies are equal to the number of vertices. This reminds us that we can think over the problem of borderenergetic graphs in view of maximum and minimum degrees. Moreover, chemical graphs also have the restriction on the maximum degrees, i.e., their maximum degrees are at most 4. So, results on graphs with small maximum degrees would have some chemical applications.

After the preliminary Section 2, we show in Section 3 that there is no borderenergetic graph with maximum degree at most 3. We then provide five necessary conditions for borderenergetic graphs with maximum degree 4, and as a result, we show that there is no borderenergetic graph with maximum degree 4 and order $n \geq 22$. So, a borderenergetic graph with maximum degree 4 must have an order n such that $n \leq 21$. Unfortunately, we cannot search out all of them because we do not have fast enough computers. In Section 4, we consider a problem contrary to that in Section 3, i.e., borderenergetic graphs with large minimum degrees. We show that there is no borderenergetic graph of order n with minimum degree $n-2$. We then construct two families of borderenergetic graphs with minimum degree $n-3$ and $n-4$, respectively, the former is for all integers $n \geq 7$ while the latter is for all even numbers $n \geq 8$.

2 Preliminaries

The following are some elementary results on the spectra or energy of graphs, which will be used in the sequel.

Lemma 2.1 *[16] For an (n,m)-graph G,*

$$\mathcal{E}(G) \leq \sqrt{2mn}$$

with equality if and only if G is either an empty graph or a regular graph of degree 1, i.e., $G \cong (n/2)K_2$.

Lemma 2.2 *[14] Let G be an (n,m)-graph. If $2m \geq n$, then*

$$\mathcal{E}(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}.$$

Moreover, equality holds if and only if G consists of $n/2$ copies of K_2, or $G \cong K_n$ or G is a noncomplete connected strongly regular graph with two nontrivial eigenvalues both having absolute values equal to $\sqrt{2m - (2m/n)^2/(n-1)}$.

Lemma 2.3 *[14] For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ where both A and B are square matrices, we have*

$$\sum_j s_j(A) + \sum_j s_j(B) \leq \sum_j s_j(C).$$

Equality holds if and only if there exist unitary matrices U and V such that the matrix $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ *is positive semidefinite, where $s_j(M)$ denotes the singular values of matrix M.*

Lemma 2.4 *[4] Let $k \geq 2$, and let Γ be a k-regular graph of order n. Then the energy per vertex of Γ (the average energy $\frac{\mathcal{E}(\Gamma)}{n}$) is at most*

$$\frac{k + (k^2 - k)\sqrt{k-1}}{k^2 - k + 1},$$

with equality if and only if Γ is the disjoint union of incidence graphs of projective planes of order $k - 1$ or, in case $k = 2$, the disjoint union of triangles and hexagons. For $n = q^2+q+1$, let π be a finite projective plane of order q with point set $P = \{p_1, \cdots, p_n\}$ and line set $L = \{l_1, \cdots, l_n\}$. A bipartite graph G with partitions (P, L) is said to be the incidence point-line graph of the projective plane π if for all i, $j \in \{1, \cdots, n\}$, p_il_j is an edge of G if and only if $p_i \in l_j$.

Lemma 2.5 *[1] (i) A graph Γ is bipartite if and only if for each eigenvalue θ of Γ, $-\theta$ is also an eigenvalue, with the same multiplicity.*

(ii) If Γ is connected with largest eigenvalue θ_1, then Γ is bipartite if and only if $-\theta_1$ is an eigenvalue of Γ.

Lemma 2.6 *[1] Let Γ be a connected graph with largest eigenvalue θ_1. If Γ is regular of valency k, then $\theta_1 = k$. Otherwise, we have $k_{min} < \overline{k} < \theta_1 < k_{max}$ where k_{min}, k_{max} and $\overline{k}$ are the minimum, maximum and average degree.*

Another little trick which will be used in our proof is stated as follows:

Lemma 2.7 *[21] We set $f_k(x_1, x_2, \cdots, x_t) = x_1^k + x_2^k + \cdots + x_t^k$, where k is a positive integer, $x_1 \geq x_2 \geq x_3 \cdots \geq x_t \geq 0$ and $\sum_{i=1}^{t} x_i = m$. If $x_i - x_j \geq 2\alpha > 0$ for some i and j, then for $k \geq 2$ we have*

$$f_k(x_1, x_2, \cdots, x_i, \cdots, x_j, \cdots, x_t) > f_k(x_1, x_2, \cdots, x_i - \alpha, \cdots, x_j + \alpha, \cdots, x_t).$$

Lemma 2.8 *[6] Let $G = (V, E)$ be the given graph and for any $S \subseteq V$, we denote by $e_{min}(S)$ the minimum number of edges that need to be removed from the induced subgraph on S to make it bipartite. Also, we denote by $cut(S)$ the set of edges with one end in S and the other in $V - S$, and we define the parameter Ψ as*

$$\Psi = \min_{S \subseteq V} \frac{e_{min}(S) + |cut(S)|}{|S|}.$$

Suppose Ψ is the parameter defined as above. Then if G is d-regular, the smallest eigenvalue of its adjacency matrix A is bounded above as $\mu_1(A) \leq -d + 4\Psi$, where $\mu_1(A)$ denotes the smallest eigenvalue of the adjacency matrix A.

We denote by $m = e(\Gamma)$ the number of edges of graph Γ, i.e, the size of Γ, and $n = |\Gamma|$ the order of Γ, similarly hereafter. It is well known that for the graph Γ, the sum of its eigenvalues is equal to the trace of Γ, that is, $\sum_{i=1}^{n} \lambda_i = 0$, and the sum of the square of its eigenvalues is equal to two times of its size, that is, $\sum_{i=1}^{n} \lambda_i^2 = 2m$, which is an essential tool in the proof.

3 Borderenergetic chemical graphs

This section is to consider borderenergetic graphs with small maximum degrees, i.e., $\Delta \leq 4$. This kind of graphs are also addressed as chemical graphs. Our results are as follows.

Theorem 3.1 *There is no noncomplete borderenergetic graph with maximum degree $\Delta = 2$ or 3.*

Proof. We distinguish two cases according to different maximum degrees.

Case 1. Let K be a borderenergetic (n, m)-graph with maximum degree $\Delta = 2$. Then $2m \leq 2n$. We only need to make use of Lemma 2.1, after simple calculation, we obtain that $\mathcal{E}(K) \leq \sqrt{2}n$. For $n \geq 4$, $\sqrt{2}n < 2n - 2$, then certainly there is no borderenergetic graphs in this scope. Since $\Delta = 2$, $|K| \geq 3$, so we only need to examine P_3. The energy of P_3 is $2\sqrt{2}$, which do not meet our expectation, and the proof of this part is done.

Case 2. Let N be a borderenergetic (n, m)-graph with maximum degree $\Delta = 3$. Then $2m \le 3n$. Similarly we use Lemma 2.1 and get $\mathcal{E}(N) \le \sqrt{3}n$. For $n \ge 8$, $\sqrt{3}n < 2n - 2$, so we only need to examine graphs with order $4 \le n \le 7$. As the spectrum of a graph is the union of the spectrum of all its connected components, so we first suppose the graph N we considered is connected and then $2m \ge n$. As the inequality in Lemma 2.2, $F(m) = \frac{2m}{n} + \sqrt{(n-1)[2m - (\frac{2m}{n})^2]}$ is an increasing function in the variable $m \in [0, \frac{3n}{2}]$. Set $m = \frac{3}{2}n$, we obtain that $\mathcal{E}(N) \le 3 + \sqrt{(n-1)(3n-9)}$, which is strictly less than $2n-2$ except for the situation $n = 4$, whereas $n = 4$ suggests that it is a complete graph so we now finish the circumstance when N is connected. If N is not connected, there certainly exists a connected component J with energy no less than $2|J| - 2$. According to the above discussion, none of these graphs exist so the analysis is done.

Theorem 3.2 *(1) Let G be a noncomplete borderenergetic graph of order n with $\Delta = 4$. Then G must have the following properties:*

(i) $e(G) = 2n$ or $2n - 1$;

(ii) $|G| \le 21$;

(iii) G is non-bipartite;

(iv) the nullity, i.e., the multiplicity of eigenvalue 0, of G is 0.

(2) Let G be a 4-regular noncomplete borderenergetic graph of order n and H is a maximal bipartite subgraph of G. Then $e(G) - e(H) \ge 3$.

We will successively prove the four properties of borderenergetic graph G with maximum degree 4. Obviously, we have $n \ge 5$.

Proof of (i): Since G has maximum degree 4, it has size $m \le 2n$. By Lemma 2.2, we have $2n - 2 \le \frac{2m}{n} + \sqrt{(n-1)[2m - (\frac{2m}{n})^2]}$. This inequality implies that $m \in \{2n-2, 2n-1, 2n\}$ and the equality holds when $m = 2n-2$. However, a borderenergetic graph satisfying the conditions listed in Lemma 2.2 is no doubt impossible to exist because the borderenergetic graph G with $2n - 2$ edges can neither be $n/2$ copies of K_2 nor a non-complete connected strongly regular graph. Hence $m \in \{2n-1, 2n\}$ and

G is either 4-regular or have size one smaller than 4-regular graph.

Proof of (ii): Firstly we deal with the case when G is 4-regular. Substitute 4 for k in the inequality of Lemma 2.4, we get that the energy per vertex in G is no more than $\frac{12\sqrt{3}+4}{13}$, which implies that $\mathcal{E}(G) \leq \frac{12\sqrt{3}+4}{13}n$. Since $\frac{12\sqrt{3}+4}{13}n < 2n-2$ when $n \geq 22$, naturally $|G| \leq 21$. As for the non-regular graphs with $2n-1$ edges, they must have all vertices of degree 4 except for two vertices of degree 3 or one vertex of degree 2. Thus we distinguish borderenergetic graphs of size $2n-1$ into two subclasses to analyze.

Suppose G is a borderenergetic (n,m)-graph with all vertices of degree 4 except for two vertices of degree 3. Take a copy of G and denote it by G'. Let u and v be the two vertices of degree 3 in G and the corresponding ones in G' are called u' and v', respectively. As is depicted in Figure 3.1, we join an edge between u and u', v and v', respectively. Clearly we have constructed a new 4-regular graph called P with G and G' being its two disjoint induced subgraph. According to Lemma 2.3, we have $\mathcal{E}(G)+\mathcal{E}(G') \leq \mathcal{E}(P) \leq \frac{12\sqrt{3}+4}{13}(2n)$. As a result, $\mathcal{E}(G) \leq \frac{12\sqrt{3}+4}{13}n$. Applying the above analysis we know that graphs with all vertices of degree 4 except for two vertices of degree 3 can also be limited to the same scope with 4-regular graphs.

Suppose G is a borderenergetic (n,m)-graph with all vertices of degree 4 except for one vertex of degree 2 denoted by ω. Similarly we take a copy of G denoted by G' with the corresponding vertex ω'. Following the above procedure we join an edge between ω and ω' (see Figure 3.1). Then we obtain a new graph Q of order $2n$ with all vertices of degree 4 except for two vertices of degree 3. The above analysis suggests $\mathcal{E}(Q) \leq \frac{12\sqrt{3}+4}{13}(2n)$ and then $\mathcal{E}(G) \leq \frac{12\sqrt{3}+4}{13}n$. Therefore, we come to a conclusion that $|G| \leq 21$ in whatever cases.

Proof of (iii): We can notice from Lemma 2.7 that if the sum of a group of positive numbers is a constant, then the more average these numbers are, the smaller the sum of their square can be. Here we consider the singular values $\{s_1, s_2, \cdots, s_n\}$ of G instead of $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and denote by S^+ the sum of all positive eigenvalues apart from λ_1, S^- the sum of all absolute value of non-positive eigenvalues except λ_n. Next we will come to the property that G is non-bipartite by contradiction. Inspired from (i), we consider this part depending on $e(G) = 2n$, $2n-1$, respectively.

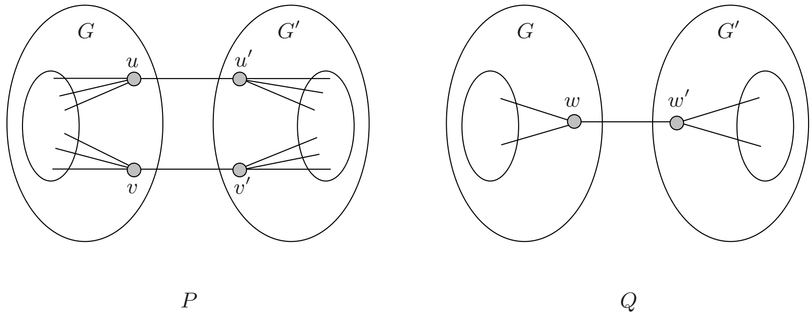

Figure 3.1: Two classes of graphs constructed in the proof of (ii)

Firstly, we assume that G is a 4-regular bipartite borderenergetic (n, m)-graph. Obviously, we have $m = 2n$ and $\lambda_1(G) = 4$. According to Lemma 2.5, $\lambda_n(G) = -4$. As $\sum_{k=1}^{n} \lambda_k(G) = 0$ and $\mathcal{E}(G) = 2n - 2$, the sum of all positive eigenvalues of G is $n - 1$ and the sum of all negative eigenvalue is $-n+1$, so $S^+(G) = S^-(G) = n-5$. We then distribute these sums such that the singular values are as average as possible. For n even, surely G have singular eigenvalue 4 with multiplicity 2, $\frac{2n-10}{n-2}$ with multiplicity $n - 2$, and then $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} s_i^2 = \frac{4(n-5)^2}{n-2} + 32$ which is strictly greater than $2m = 4n$, a contradiction. We claim that there is no (X, Y)-bipartite regular graph with odd order. Suppose not, since the sums of the degrees of vertices in X and Y are equal which implies $|X| = |Y|$, we obtain that $n = 2|X| = 2|Y|$ is even, a contradiction.

Suppose G is a bipartite borderenergetic graph of order n and size $m = 2n - 1$. We adopt the same method as the above situation. According to Lemma 2.6, $\lambda_1(G) > \frac{2m}{n} = 4 - \frac{2}{n}$. In order to achieve our goal of distributing the singular values as average as possible, we set $\lambda_1(G) = 4 - \frac{2}{n}$ and $\lambda_n(G) = -4 + \frac{2}{n}$. When n is even, similar to the way we distribute above, $\sum \lambda_i^2 = \sum s_i^2 > \frac{4(n-5+\frac{2}{n})^2}{n-2} + 2(4-\frac{2}{n})^2$, which is strictly greater than $2m = 4n - 2$, a contradiction. Next we claim that there is no (X, Y)-bipartite graph with odd order n and size $2n - 1$. If not, we denote by j the sum of degrees of vertices in X while l in Y. Then it is well known that $j = l = 2n - 1$. Consider that G have all vertices of degree 4 except for a vertex of degree 2 or two vertices of degree 3. In the former case, without loss of generality, assume that the vertex of degree 2 is in X, then $j = 2 \pmod 4 \neq 0 \pmod 4 = l$, a contradiction. While in the latter case, X and Y must each contains one vertex of degree 3 and the same number of vertices of degree 4 which imply that $|X| = |Y|$, also a contradiction.

Proof of (iv): Firstly, we assume that G is a 4-regular borderenergetic (n, m)-graph containing 0 as its eigenvalue with multiplicity k. Then on the basis of the rule of "average", we figure out the possible smallest value for $\sum_{i=1}^{n} \lambda_i^2$ is when G has eigenvalue 4 with multiplicity 1 and 0 with multiplicity k, and all others with absolute value $\frac{2n-6}{n-k-1}$. Hence in this case $\sum_{i=1}^{n} \lambda_i^2 = \frac{4(n^2-6n+9)}{n-k-1} + 16$, which is strictly greater than $2m = 4n$ when $k \geq 1$, a contradiction. Thus we come to a conclusion that the nullity of G is 0. Next we assume that G is non-regular, again Lemma 2.6 suggests that $\lambda_1(G) > \frac{2m}{n} = 4 - \frac{2}{n}$. Then the most average distribution is eigenvalue $4 - \frac{2}{n}$ with multiplicity 1, 0 with multiplicity k and all others with absolute value $\frac{2n-6+\frac{2}{n}}{n-k-1}$, while G can not reach this. Then $\sum_{i=1}^{n} \lambda_i^2 > (4 - \frac{2}{n})^2 + \frac{(2n-6+\frac{2}{n})^2}{n-k-1} = 4n - 2 + \frac{2}{n-k-1}\left[2nk - n - 9k - \frac{2}{n} + \frac{8k}{n} - \frac{2k}{n^2} + 5\right]$. Since $0 < \frac{2k}{n^2} < 1$, then $4n - 2 + \frac{2}{n-k-1}\left[2nk - n - 9k - \frac{2}{n} + \frac{8k}{n} - \frac{2k}{n^2} + 5\right] > 4n - 2 + \frac{2}{n-k-1}\left[2nk - n - 9k + 4 + \frac{8k-2}{n}\right]$. Set $f(k) = 2nk - n - 9k + 4 + \frac{8k-2}{n}$, then $f'(k) = 2n - 9 + \frac{8}{n} > 0$ and so $f(k)$ is an increasing function on $k \geq 1$. Therefore, $f(k) \geq f(1) = n - 5 + \frac{6}{n} > 0$ and thus $\sum_{i=1}^{n} \lambda_i^2 > 4n-2+\frac{2}{n-k-1}\left[2nk-n-9k+4+\frac{8k-2}{n}\right] > 4n-2$, which contradicts to that $\sum_{i=1}^{n} \lambda_i^2 = 2m = 4n - 2$. To sum up, G must have nullity 0 in all cases we talk about.

Proof of (2): For a 4-regular borderenergetic (n, m)-graph G with $\lambda_n(G) = x$, the most average distribution for the eigenvalues of G is to have eigenvalue 4 with multiplicity 1, x with multiplicity 1 and all others with absolute value $\frac{2n-6+x}{n-2}$. Then we have $\frac{(2n-6+x)^2}{n-2}+x^2+16 \leq \sum_{i=1}^{n} \lambda_i^2 = 2m = 4n$. Solve this inequality and we obtain that $x \in \left(\frac{6-2n-2\sqrt{n^2-7n+10}}{n-1}, \frac{6-2n+2\sqrt{n^2-7n+10}}{n-1}\right)$. Obviously, $x = \frac{6-2n-2\sqrt{n^2-7n+10}}{n-1}$ is absolutely the smallest possible value for x. According to Lemma 2.8, $\frac{6-2n-2\sqrt{n^2-7n+10}}{n-1} \leq -4+4\Psi$, whereas $\Psi \leq \frac{e_{min}(V)+|cut(V)|}{|V|} = \frac{e_{min}(V)}{n}$, hence $\frac{6-2n-2\sqrt{n^2-7n+10}}{n-1} \leq -4 + \frac{4e_{min}(V)}{n}$. Solve this inequality and we get $e_{min}(V) > \frac{9}{4}+\frac{11}{4(n-4)}$, which suggests that $e_{min}(V) \geq 3$. This suggests that $e(G) - e(H) \geq 3$.

The proof is now complete. ∎

Remark 1 *The noncomplete borderenergetic graphs of order $n \leq 11$ can be singled out from [8, 15, 20]. There are only two of them with maximum degree $\Delta = 4$; see*

G_1, G_2 (depicted in Figure 3.2). In addition, Theorem 3.2 implies that a noncomplete borderenergetic graph with $\Delta = 4$ except for G_1, G_2 must be non-bipartite and has an order n such that $12 \le n \le 21$, and a size m such that $m = 2n$ or $2n - 1$, and has nullity 0. Unfortunately, we cannot completely give all noncomplete borderenergetic graphs with $\Delta = 4$ because our computers are not fast enough to search out all of them. The reader(s) can try to search them with some better computers and computing techniques.

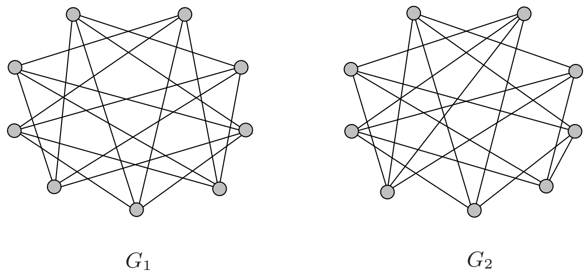

Figure 3.2: Two noncomplete borderenergetic graphs of order $n \le 11$ and $\Delta = 4$.

4 Borderenergetic graphs with $\delta \ge n - 4$

The above section only considered borderenergetic graphs with small maximum degrees, which has applications for chemical graphs. This section is to consider borderenergetic graphs with large minimum degrees.

Lemma 4.1 *[12] Let $A \in \mathcal{M}_n^0$, $n \ge 2$, where $\mathcal{M}_n^0 = \{A = (a_{ij}) \in \mathbb{R}^n : A = A^T, 0 \le a_{ij} \le 1$ and $a_{ii} = 0$ for $1 \le i, j \le n,\}$ Set $A' = J_n - I_n - A$, where J_n denotes the $n \times n$ matrix of all ones and I_n the identity matrix of order n. Then*

$$\lambda_2(A) \le \lambda_1(A') - 1.$$

Furthermore, equality holds if and only if either $A = J_n - I_n$, i.e, $A' = 0$, or, for a certain nonempty subset $K \subseteq \langle n \rangle = \{1, 2, \cdots, n\}$, the following conditions are fulfilled: (i) $A'[K]$ is an irreducible component of A', where $A'[K]$ denotes the principle submatrix of A' with row and column indices in K;

(ii) $\lambda_1(A'[K]) = \lambda_1(A')$;

(iii) $A'[K]$ *is a* 2*-cyclic matrix, i.e, for a certain nonempty subset* $S \subseteq \langle n \rangle$ *and* $\overline{S} = \langle n \rangle \backslash S$, *then we have* $A[S] = \mathbf{0}$ *and* $A[\overline{S}] = \mathbf{0}$;

(iv) there is a nonzero vector $x \in \mathbb{R}^n$ *such that*

$$A'x = \lambda_n(A')x \quad and \quad e^T x = 0$$

where $e = [1, 1, \cdots, 1]^T \in \mathbb{R}^n$.

Lemma 4.2 *[10] Let* G *be a* k*-regular graph of order* n *with spectrum* $Sp(G) = \{k, \lambda_2, \cdots, \lambda_n\}$. *Then the spectrum of the complement* $\overline{G}$ *of* G *is* $Sp(\overline{G}) = \{n - 1 - k, -1 - \lambda_2, \cdots, -1 - \lambda_n\}$.

Lemma 4.3 *[8] Let* p, q *and* r *be non-negative integers, and let* $p + q = 2$. *Then* $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$ *is borderenergetic.*

Lemma 4.4 *[8] The smallest noncomplete borderenergetic graph has* $n = 7$ *vertices and is unique. There exist (exactly) six noncomplete borderenergetic graphs of order* 8, *five of which have minimum degree* $\delta = 4 = n - 4$ *and one noncomplete borderenergetic graph of order* 9 *with minimum degree* $\delta = 6 = n - 3$.

Theorem 4.5 *No borderenergetic graphs have minimum degree* $n-2$. *Besides, for each integer* $n \geq 7$, *there exists a connected noncomplete borderenergetic graph of order* n *with minimum degree* $n-3$ *and for each even integer* $n \geq 8$, *there exists a noncomplete borderenergetic graph of order* n *with minimum degree* $n - 4$.

Proof. We distinguish three cases according to different minimum degrees.

Case 1. Let G be a borderenergetic (n, m)-graph with minimum degree $n - 2$. Applying Lemma 2.6, we have $n - 2 \leq \lambda_1(G) < n - 1$ because its maximum degree can not exceed $n - 1$. Taking Lemma 4.1 into consideration, we know that $\lambda_2(G) \leq \lambda_1(\overline{G})-1$. Since G has minimum degree $n-2$, then $\overline{G}$ must be a matching. Additionally, a matching has half of its eigenvalues 1 and the other half -1, so we have $\lambda_2(G) \leq 0$.

It is well known that $\sum_{i=1}^{n}\lambda_i(G)=0$, so the sum of the absolute values for all positive and negative eigenvalues should be equal, that is, the sum of all positive values is $n-1=\lambda_1(G)$, which is a contradiction.

Case 2. What the construction given in Lemma 4.3 produces are all $(n-3)$-regular graphs. Notice that the case for $p=1$ and $q=1$ constructs graphs with order $n\geq 10$ and $n=1 \pmod 3$, $p=2$ and $q=0$ contributes to graphs with order $n\geq 8$ and $n=2 \pmod 3$, $p=0$ and $q=2$ obtains graphs with order $n\geq 12$ and $n=0 \pmod 3$. From Lemma 4.4 we know that for orders 7 and 9, there also exists graphs satisfying the demands, and then we finish this part.

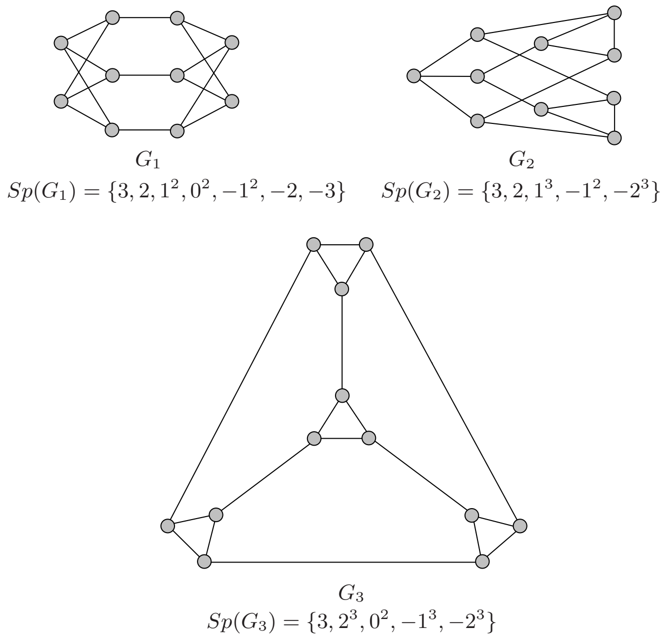

Figure 4.3: Graphs for Theorem 4.5.

Case 3. Now we concentrate our minds on constructing $(n-4)$-regular borderenergetic integral graphs. Similar to our discussion above, G must have sum of all its positive eigenvalues $n-1$, whereas it has $\lambda_1=n-4$ so the addition of other positive eigenvalues is 3. Obviously $\overline{G}$ is a union of cubic graphs and it must have the smallest eigenvalue -2 with multiplicity 3 or eigenvalues -2 and -3 both with multiplicity 1. Referring to [19], we know that there are three integral cubic graphs having the above property. These three graphs are depicted in Figure 4.3, denoted by G_1, G_2 and G_3. Easy to see that $|V(G_1)|=|V(G_2)|=10$ and $|V(G_3)|=12$. Now we construct a class of new graphs as $G = \overline{pG_1 \bigcup qG_2 \bigcup rG_3 \bigcup sK_4}$, where p, q, r and s are all non-negative integers and $p + q + r = 1$. One can easily check that G is borderenergetic.

Put $r = 0$ and $p = 1$ or $q = 1$, we construct borderenergetic graphs with order $n \geq 10$ and $n = 2 \pmod 4$, the condition of $r = 1$, $p = q = 0$ create borderenergetic graphs with order $n \geq 12$ and $n = 0 \pmod 4$. According to Lemma 4.4, borderenergetic graph of order 8 with desired property also exists. Graphs of order 6 can certainly be ignored since the smallest noncomplete borderenergetic graph has order 7 due to Lemma 4.4. Thus the proof is complete. ∎

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