

# $s$ -Inversion Sequences and $P$ -Partitions of Type $B$

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## Abstract

Given a sequence  $s = (s_1, s_2, \dots)$  of positive integers, the notion of inversion sequences with respect to  $s$ , or  $s$ -inversion sequences, was introduced by Savage and Schuster in their study of lecture hall polytopes. A sequence  $(e_1, e_2, \dots, e_n)$  of nonnegative integers is called an  $s$ -inversion sequence of length  $n$  if  $0 \leq e_i < s_i$  for  $1 \leq i \leq n$ . Let  $I_n$  be the set of  $s$ -inversion sequences of length  $n$  for  $s = (1, 4, 3, 8, 5, 12, \dots)$ , that is,  $s_{2i} = 4i$  and  $s_{2i-1} = 2i - 1$  for  $i \geq 1$ , and let  $P_n$  be the set of signed permutations on the multiset  $\{1^2, 2^2, \dots, n^2\}$ . Savage and Visontai conjectured that the descent number over  $P_n$  is equidistributed with the ascent number over  $I_{2n}$ . In this paper, we give a proof of this conjecture by using  $P$ -partitions of type  $B$ . We notice that an independent proof based on recurrence relations was found by Lin. Moreover, we find a set of signed permutations over which the descent number is equidistributed with the ascent number over  $I_{2n-1}$ . Let  $I'_n$  be the set of  $s$ -inversion sequences of length  $n$  for  $s = (2, 2, 6, 4, 10, 6, \dots)$ , that is,  $s_{2i} = 2i$  and  $s_{2i-1} = 4i - 2$  for  $i \geq 1$ . We also find two sets of signed permutations over which the descent number is equidistributed with the ascent number over  $I'_n$ , depending on whether  $n$  is even or odd.

**Keywords:** inversion sequence, ascent number, signed permutation, descent number,  $P$ -partition of type  $B$

**AMS Subject Classifications:** 05A05, 05A15

# 1 Introduction

The notion of  $s$ -inversion sequences was introduced by Savage and Schuster [4] in their study of lecture hall polytopes. Let  $s = (s_1, s_2, \dots)$  be a sequence of positive integers. An *inversion sequence* of length  $n$  with respect to  $s$ , or an  $s$ -inversion sequence of length  $n$ , is a sequence  $e = (e_1, e_2, \dots, e_n)$  of nonnegative integers such that  $0 \leq e_i < s_i$  for  $1 \leq i \leq n$ . An *ascent* of an  $s$ -inversion sequence  $e = (e_1, e_2, \dots, e_n)$  is defined to be an integer  $i \in \{0, 1, \dots, n-1\}$  such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

where we assume that  $e_0 = 0$  and  $s_0 = 1$ . The ascent number  $\text{asc}(e)$  of  $e$  is meant to be the number of ascents of  $e$ .

The generating function of ascent numbers of  $s$ -inversion sequences can be viewed as a generalization of the Eulerian polynomial for permutations, since the ascent number over  $s$ -inversion sequences of length  $n$  for  $s = (1, 2, 3, \dots)$  is equidistributed with the descent number over permutations on  $\{1, 2, \dots, n\}$ , see Savage and Schuster [4]. Savage and Visontai [5] showed that for any sequence  $s$  of positive integers and any positive integer  $n$ , the generating function of ascent numbers of  $s$ -inversion sequences of length  $n$  has only real roots. In particular, by establishing a relation between the generating function of ascent numbers of  $s$ -inversion sequence for  $s = (2, 4, 6, \dots)$  and the generating function of descent numbers of even-signed permutations, they proved the real-rootedness of the Eulerian polynomial of type  $D$  as conjectured by Brenti [1].

Savage and Visontai [5] also found that the descent number over permutations on the multiset  $\{1^2, 2^2, \dots, n^2\}$ , where  $i^2$  stands for two occurrences of  $i$  for  $1 \leq i \leq n$ , is equidistributed with the ascent number over  $s$ -inversion sequences of length  $2n$  with  $s = (1, 1, 3, 2, 5, 3, \dots)$ . They further conjectured that the descent number over signed permutations on  $\{1^2, 2^2, \dots, n^2\}$  is equidistributed with the ascent number over  $s$ -inversion sequences of length  $2n$  with  $s = (1, 4, 3, 8, 5, 12, \dots)$ . Let  $P_n$  denote the set of signed permutations on  $\{1^2, 2^2, \dots, n^2\}$ , and let  $I_n$  denote the set of  $s$ -inversion sequences of length  $n$  for  $s = (1, 4, 3, 8, 5, 12, \dots)$ . Savage and Visontai [5] posed the following conjecture, which implies the real-rootedness of the generating function of descent numbers of signed permutations in  $P_n$ .

**Conjecture 1.1** (Savage and Visontai [5]) *For  $n \geq 1$ , the descent number over  $P_n$  is equidistributed with the ascent number over  $I_{2n}$ .*

In this paper, we give a proof of Conjecture 1.1. Let  $P_n(x)$  denote the generating function of descent numbers of signed permutations in  $P_n$ , and let  $I_n(x)$  denote the generating function of ascent numbers of inversion sequences in  $I_n$ . Savage and Schuster [4] deduced that

$$\frac{I_{2n}(x)}{(1-x)^{2n+1}} = \sum_{t \geq 0} (t+1)^n (2t+1)^n x^t. \quad (1.1)$$

Using  $P$ -partitions of type  $B$  introduced by Chow [2], we show that  $P_n(x)$  satisfies the same relation as  $I_{2n}(x)$ . Thus  $P_n(x) = I_{2n}(x)$ , and this proves Conjecture 1.1.

It should be noted that Lin [3] found another proof of Conjecture 1.1 by proving that the coefficients of  $P_n(x)$  and  $I_{2n}(x)$  satisfy the same recurrence relation.

Besides the equidistribution conjectured by Savage and Visontai, we also find a set of signed permutations over which the descent number is equidistributed with the ascent number over  $I_{2n-1}$ . Let  $U_n(x)$  be the generating function of descent numbers of signed permutations on  $\{1^2, 2^2, \dots, (n-1)^2, n\}$ , and let  $V_n(x)$  be the generating function of descent numbers of signed permutations on  $\{1^2, 2^2, \dots, (n-1)^2, n\}$  in which  $n$  has a minus sign. Similar to relation (1.1) for  $I_{2n}(x)$ , Savage and Schuster [4] deduced a relation for  $I_{2n-1}(x)$ . We show that  $V_n(x)$  satisfies the same relation as  $I_{2n-1}(x)$ , and thus we obtain that  $I_{2n-1}(x) = V_n(x)$ .

Moreover, let  $I'_n$  be the set of  $s$ -inversion sequences of length  $n$  for  $s = (2, 2, 6, 4, 10, 6, \dots)$ . We use  $I'_n(x)$  to denote the generating function of ascent numbers of inversion sequences in  $I'_n$ . Using similar arguments to  $I_n(x)$ , we obtain that  $I'_{2n}(x) = P_n(x)$  and  $I'_{2n-1}(x) = U_n(x)$ .

## 2 Proof of Conjecture 1.1

In this section, we present a proof of Conjecture 1.1. For  $n \geq 1$ , we use  $F_n$  to denote the forest consisting of  $n$  rooted trees each of which has exactly two vertices. We show that the generating function  $P_n(x)$  of descent numbers of signed permutations on  $\{1^2, 2^2, \dots, n^2\}$  equals the generating function  $G_n(x)$  of descent numbers of linear extensions of  $F_n$  with signed labelings under certain conditions. Using the technique of  $P$ -partitions of type  $B$ , we deduce that  $G_n(x)$  satisfies the same relation (1.1) as  $I_{2n}(x)$ , which implies that  $G_n(x) = I_{2n}(x)$ . Thus we reach the conclusion that  $P_n(x) = I_{2n}(x)$ , and this proves Conjecture 1.1.

Let us begin with an overview of linear extensions of a poset. Let  $P$  be a poset on the set  $\{v_1, v_2, \dots, v_n\}$  with order relation  $\leq$ . As usual, we use the notation  $v_i < v_j$  to denote that  $v_i \leq v_j$  but  $v_i \neq v_j$ . A labeling of  $P$  is an assignment of positive integers to the elements  $v_1, v_2, \dots, v_n$  such that each positive integer cannot be used more than once. A signed labeling of  $P$  is a labeling of  $P$  with each label possibly associated with a minus sign. We adopt the notation  $(P, w)$  for a signed labeled poset, where  $w$  is a signed labeling of  $P$ . For a signed labeled poset  $(P, w)$  and an element  $v$  of  $P$ , we use  $w(v)$  to denote the label associated with  $v$ .

In this paper, we will be concerned only with a special type of posets, namely, labeled forests with each tree consisting of at most two vertices. Such a forest will be called a *simple forest*. When viewed as a poset, a simple forest is endowed with the following order relation. We say that  $u < v$  if  $u$  is a child of  $v$ . For example, Figure 2.1 illustrates

a simple forest  $P$  along with a signed labeling of  $P$ .

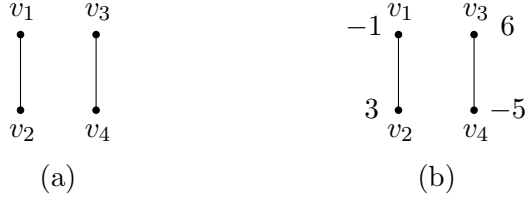


Figure 2.1: A simple forest along with a signed labeling.

Recall that a *linear extension* of a poset  $P$  is a permutation  $v_{i_1}v_{i_2}\cdots v_{i_n}$  of the elements of  $P$  such that  $v_{i_j} < v_{i_k}$  only if  $j < k$ , see Stanley [6]. However, by a linear extension of a signed labeled poset  $(P, w)$  we mean a permutation  $w(v_{i_1})w(v_{i_2})\cdots w(v_{i_n})$  of the labels associated with the elements of  $P$ , where  $v_{i_1}v_{i_2}\cdots v_{i_n}$  is a linear extension of  $P$ . Let  $\mathcal{L}(P, w)$  denote the set of linear extensions of  $(P, w)$ . For example, for the signed labeled forest in Figure 2.1, we have

$$\mathcal{L}(P, w) = \{3\bar{1}\bar{5}6, 3\bar{5}\bar{1}6, 3\bar{5}6\bar{1}, \bar{5}3\bar{1}6, \bar{5}36\bar{1}, \bar{5}63\bar{1}\},$$

where  $\bar{i}$  is identified with  $-i$ .

In this section, we shall further restrict our attention to simple forests for which each component is a rooted tree with two vertices. More precisely, let  $F_n$  denote such a simple forest with  $n$  trees  $T_1, T_2, \dots, T_n$ , where  $T_i$  is rooted at  $v_i$  with  $u_i$  being the only child. A signed labeling  $w$  of  $F_n$  is said to be *local* if it satisfies one of the following conditions:

- (1)  $w(u_i) = 2i - 1$  and  $w(v_i) = 2i$ ;
- (2)  $w(u_i) = \overline{2i - 1}$  and  $w(v_i) = 2i$ ;
- (3)  $w(u_i) = 2i - 1$  and  $w(v_i) = \overline{2i}$ ;
- (4)  $w(u_i) = \overline{2i}$  and  $w(v_i) = \overline{2i - 1}$ .

We use  $L(F_n)$  to denote the set of local signed labelings of  $F_n$ . A linear extension of  $F_n$  with a local signed labeling becomes a signed permutation on  $\{1, 2, \dots, 2n\}$ . A signed permutation on  $\{1, 2, \dots, 2n\}$  is a permutation on  $\{1, 2, \dots, 2n\}$  for which each element is possibly associated with a minus sign. A signed permutation on a multiset  $\{1^2, 2^2, \dots, n^2\}$  can be defined in the same manner. As will be shown in Theorem 2.1, the generating function  $P_n(x)$  of descent numbers of signed permutations on the multiset  $\{1^2, 2^2, \dots, n^2\}$  equals the generating function  $G_n(x)$  of descent numbers of linear extensions of  $F_n$  with local signed labelings.

Recall that the descent set of a signed permutation  $\sigma = \sigma_1\sigma_2\cdots\sigma_n$  is defined as

$$\{i \mid \sigma_i > \sigma_{i+1}, 1 \leq i \leq n - 1\} \cup \{0 \mid \text{if } \sigma_1 < 0\}, \quad (2.1)$$

see Savage and Visontai [5]. However, for the purpose of this paper, we choose the following alternative definition of the descent set of  $\sigma$ :

$$\{i \mid \sigma_i > \sigma_{i+1}, 1 \leq i \leq n-1\} \cup \{n \mid \sigma_n > 0\}. \quad (2.2)$$

The descent number  $\text{des}_B(\sigma)$  of  $\sigma$  is referred to as the number of elements in the descent set defined by (2.2). In fact, via the bijection

$$\sigma = \sigma_1\sigma_2 \cdots \sigma_n \quad \mapsto \quad \sigma' = (-\sigma_n)(-\sigma_{n-1}) \cdots (-\sigma_1),$$

we see that the descent numbers defined by (2.1) and (2.2) are equidistributed over the set of signed permutations on  $\{1^2, 2^2, \dots, n^2\}$ .

With the above notation, the generating function  $G_n(x)$  can be written as

$$G_n(x) = \sum_{w \in L(F_n)} \sum_{\sigma \in \mathcal{L}(F_n, w)} x^{\text{des}_B(\sigma)}.$$

We have the following equidistribution property.

**Theorem 2.1** *For  $n \geq 1$ , we have*

$$G_n(x) = P_n(x).$$

*Proof.* Define a map  $\phi$  from the set

$$\bigcup_{w \in L(F_n)} \mathcal{L}(F_n, w) \quad (2.3)$$

of linear extensions of  $F_n$  with local signed labelings to the set of signed permutations on  $\{1^2, 2^2, \dots, n^2\}$ . Let  $\sigma = \sigma_1\sigma_2 \cdots \sigma_{2n}$  be a linear extension in  $\mathcal{L}(F_n, w)$ , where  $w \in L(F_n)$ . The construction of  $\phi(\sigma) = \tau = \tau_1\tau_2 \cdots \tau_{2n}$  can be described as follows. For  $1 \leq i \leq 2n$ , assume that  $\tau_i$  has the same sign as  $\sigma_i$ . Moreover, set  $|\tau_i| = \frac{|\sigma_i|}{2}$  if  $|\sigma_i|$  is even and set  $|\tau_i| = \frac{|\sigma_i|+1}{2}$  if  $|\sigma_i|$  is odd. Since  $\sigma$  is a signed permutation on  $\{1, 2, \dots, 2n\}$ , it can be easily checked that  $\tau$  is a signed permutation on  $\{1^2, 2^2, \dots, n^2\}$ .

To show that  $\phi$  is a bijection, we construct a map  $\psi$  from the set of signed permutations on  $\{1, 2, \dots, 2n\}$  to the set in (2.3) and we shall prove that  $\psi$  is the inverse of  $\phi$ . Let  $\tau = \tau_1\tau_2 \cdots \tau_{2n}$  be a signed permutation on  $\{1^2, 2^2, \dots, n^2\}$ . Define  $\psi(\tau) = \sigma = \sigma_1\sigma_2 \cdots \sigma_{2n}$  by the following procedure. For each  $1 \leq i \leq n$ , assume that  $a_i$  and  $b_i$  ( $a_i < b_i$ ) are the two positions of  $\tau$  occupied by  $i$  or  $\bar{i}$ . Moreover,  $\sigma_{a_i}$  and  $\sigma_{b_i}$  are determined according to the following cases:

- (1)  $\sigma_{a_i} = 2i - 1$  and  $\sigma_{b_i} = 2i$  if  $\tau_{a_i} = \tau_{b_i} = i$ ;
- (2)  $\sigma_{a_i} = \overline{2i - 1}$  and  $\sigma_{b_i} = 2i$  if  $\tau_{a_i} = \bar{i}$  and  $\tau_{b_i} = i$ ;

- (3)  $\sigma_{a_i} = 2i - 1$  and  $\sigma_{b_i} = \overline{2i}$  if  $\tau_{a_i} = i$  and  $\tau_{b_i} = \overline{i}$ ;
- (4)  $\sigma_{a_i} = \overline{2i}$  and  $\sigma_{b_i} = \overline{2i - 1}$  if  $\tau_{a_i} = \tau_{b_i} = \overline{i}$ .

So  $\sigma$  is a signed permutation on  $\{1, 2, \dots, 2n\}$ . Let  $w$  be a signed labeling of  $F_n$  defined by  $w(u_i) = \sigma_{a_i}$  and  $w(v_i) = \sigma_{b_i}$ . It is routine to check that  $w$  is a local signed labeling of  $F_n$ . It is also straightforward to verify that  $\sigma$  is a linear extension of  $(F_n, w)$ .

For any linear extension  $\sigma$  of  $F_n$  with a local signed labeling, by direct verification we see that  $\psi(\phi(\sigma)) = \sigma$ . This implies that  $\psi$  is the inverse of  $\phi$ , and hence  $\phi$  is a bijection. Finally, by the construction of  $\phi$ , it can be seen that  $j \in \{1, 2, \dots, 2n\}$  is a descent of  $\sigma$  if and only if it is a descent of  $\phi(\sigma)$ . This completes the proof.  $\blacksquare$

As the simplest example of the bijection  $\phi$ , consider the case  $n = 1$ . For  $F_1$ , there are four local signed labelings and the set of linear extensions of  $F_1$  with local signed labelings is  $\{12, \overline{1}2, 1\overline{2}, \overline{2}\overline{1}\}$ . In this case, we have

$$\phi(12) = 11, \quad \phi(\overline{1}2) = \overline{1}1, \quad \phi(1\overline{2}) = 1\overline{1}, \quad \phi(\overline{2}\overline{1}) = \overline{1}\overline{1}.$$

The next theorem shows that  $G_n(x)$  satisfies the same relation as  $I_{2n}(x)$ .

**Theorem 2.2** *For  $n \geq 1$ , we have*

$$\frac{G_n(x)}{(1-x)^{2n+1}} = \sum_{t \geq 0} (t+1)^n (2t+1)^n x^t. \quad (2.4)$$

To prove the above theorem, recall the notion of a  $(P, w)$ -partition of type  $B$  introduced by Chow [2]. Let  $P$  be a poset and  $w$  be a signed labeling of  $P$ . A  $(P, w)$ -partition of type  $B$  is a map  $f$  from  $P$  to the set of nonnegative integers that satisfies the following conditions:

- (1)  $f(u) \geq f(v)$  if  $u \leq v$ ;
- (2)  $f(u) > f(v)$  if  $u < v$  and  $w(u) > w(v)$ ;
- (3)  $f(v) \geq 1$  if  $w(v) > 0$ .

When  $w$  is a labeling with positive integers, a  $(P, w)$ -partition of type  $B$  reduces to an ordinary  $(P, w)$ -partition defined by Stanley [6]. Substituting each element  $v \in P$  with its label  $w(v)$ , a  $(P, w)$ -partition of type  $B$  can be viewed as a map from the set of labels of  $P$  to the set nonnegative integers. Chow [2] showed that  $(P, w)$ -partitions of type  $B$  can be generated by linear extensions of  $(P, w)$ . For a linear extension  $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$  of  $(P, w)$ , a map  $g$  from  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  to the set of nonnegative integers is called  $\sigma$ -compatible if the following conditions are satisfied:

- (1)  $g(\sigma_1) \geq g(\sigma_2) \geq \cdots \geq g(\sigma_n)$ ;
- (2)  $g(\sigma_i) > g(\sigma_{i+1})$  if  $1 \leq i \leq n - 1$  and  $\sigma_i > \sigma_{i+1}$ ;
- (3)  $g(\sigma_n) \geq 1$  if  $\sigma_n > 0$ .

Notice that for two distinct linear extensions  $\sigma$  and  $\sigma'$  of  $(P, w)$ , any  $\sigma$ -compatible map is not  $\sigma'$ -compatible.

The following theorem is due to Chow [2], which will be used to establish a relation between the generating function for the number of  $(P, w)$ -partitions of type  $B$  and the generating function for the descent number of linear extensions of  $(P, w)$ .

**Theorem 2.3** (Chow [2]) *Let  $P$  be a poset with a signed labeling  $w$ . A map  $f$  from  $P$  to the set of nonnegative integers is a  $(P, w)$ -partition of type  $B$  if and only if there exists a linear extension  $\sigma$  of  $(P, w)$  such that  $f$  is  $\sigma$ -compatible.*

For a nonnegative integer  $t$ , let  $\Omega_P(w, t)$  denote the number of  $(P, w)$ -partitions  $f$  of type  $B$  such that  $f(v) \leq t$  for any  $v \in P$ . We have the following relation.

**Theorem 2.4** *Let  $P$  be a poset with  $n$  elements, and let  $w$  be a signed labeling of  $P$ . Then*

$$\frac{1}{(1-x)^{n+1}} \sum_{\sigma \in \mathcal{L}(P, w)} x^{\text{des}_B(\sigma)} = \sum_{t \geq 0} \Omega_P(w, t) x^t. \quad (2.5)$$

*Proof.* The proof is analogous to that of Stanley [6] for the case of an ordinary labeling. For a linear extension  $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$  of  $(P, w)$ , let  $\Omega_\sigma(t)$  denote the number of  $\sigma$ -compatible maps  $g$  such that  $g(\sigma_i) \leq t$  for  $1 \leq i \leq n$ . In view of Theorem 2.3, we see that

$$\Omega_P(w, t) = \sum_{\sigma \in \mathcal{L}(P, w)} \Omega_\sigma(t).$$

Thus, to prove (2.5) it suffices to show that

$$\sum_{t \geq 0} \Omega_\sigma(t) x^t = \frac{x^{\text{des}_B(\sigma)}}{(1-x)^{n+1}}. \quad (2.6)$$

To count  $\Omega_\sigma(t)$ , we establish a bijection between the set of  $\sigma$ -compatible maps  $g$  with  $g(\sigma_i) \leq t$  for  $1 \leq i \leq n$  and the set of partitions  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  with  $\lambda_1 \leq t - \text{des}_B(\sigma)$ . Recall that a partition is a sequence  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  of nonnegative integers such that  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ . For  $1 \leq i \leq n$ , let  $d_i$  denote the number of descents of  $\sigma$  that are greater than or equal to  $i$ , that is,

$$d_i = |\{j \mid i \leq j \leq n - 1, \sigma_j > \sigma_{j+1}\} \cup \{n \mid \text{if } \sigma_n > 0\}|.$$

Let  $g$  be a  $\sigma$ -compatible map with  $g(\sigma_i) \leq t$  for  $1 \leq i \leq n$ . It is easily checked that by setting  $\lambda_i = g(\sigma_i) - d_i$  for  $1 \leq i \leq n$ , we are given a partition  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  with  $\lambda_1 \leq t - \text{des}_B(\sigma)$ . It can be seen that this procedure is reversible. So we arrive at a bijection. Notice that the number of partitions  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  with  $\lambda_1 \leq t - \text{des}_B(\sigma)$  is equal to

$$\binom{n + t - \text{des}_B(\sigma)}{n},$$

see Stanley [6]. It follows that

$$\Omega_\sigma(t) = \binom{n + t - \text{des}_B(\sigma)}{n},$$

which implies (2.6). This completes the proof.  $\blacksquare$

We are now ready to prove Theorem 2.2.

*Proof of Theorem 2.2.* By Theorem 2.4, we aim to prove the following equivalent form of (2.4):

$$\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = ((t+1)(2t+1))^n. \quad (2.7)$$

Recall that  $F_n$  consists of  $n$  components  $T_1, T_2, \dots, T_n$ , where  $T_i$  is a tree rooted at  $v_i$  with  $u_i$  being the only child. Keep in mind that the left-hand side of (2.7) equals the number of  $(F_n, w)$ -partitions  $f$  of type  $B$  such that  $f(u_i) \leq t$  and  $f(v_i) \leq t$ , where  $w$  is a local signed labeling of  $F_n$ . Restricting  $f$  to the tree  $T_i$ , we obtain a map  $f_i$  from  $T_i$  to the set nonnegative integers. Similarly, restricting  $w$  to  $T_i$  gives a signed labeling  $w_i$  of  $T_i$ . Recall that  $w_i$  is given by one of the following assignments:

- (1)  $w_i(u_i) = 2i - 1$  and  $w_i(v_i) = 2i$ ,
- (2)  $w_i(u_i) = \overline{2i - 1}$  and  $w_i(v_i) = 2i$ ,
- (3)  $w_i(u_i) = 2i - 1$  and  $w_i(v_i) = \overline{2i}$ ,
- (4)  $w_i(u_i) = \overline{2i}$  and  $w_i(v_i) = \overline{2i - 1}$ .

Clearly,  $f_i$  is a  $(T_i, w_i)$ -partition of type  $B$  satisfying the conditions  $f_i(u_i) \leq t$  and  $f_i(v_i) \leq t$ . Conversely,  $f$  can be recovered from  $f_1, f_2, \dots, f_n$ .

For a signed labeling  $w_i$  of  $T_i$  induced by a local signed labeling of  $F_n$ , we now compute the number  $\Omega_{T_i}(w_i, t)$  of  $(T_i, w_i)$ -partitions  $f_i$  of type  $B$  such that  $f_i(u_i) \leq t$  and  $f_i(v_i) \leq t$ . We consider the above four cases.

Case 1:  $w_i(u_i) = 2i - 1$  and  $w_i(v_i) = 2i$ . It is easily seen that in this case  $f_i$  is a  $(T_i, w_i)$ -partition of type  $B$  if and only if

$$0 < f_i(v_i) \leq f_i(u_i) \leq t.$$



So we have

$$\Omega_{T_i}(w_i, t) = \binom{t+1}{2}.$$

Case 2:  $w_i(u_i) = \overline{2i-1}$  and  $w_i(v_i) = 2i$ . Similarly, in this case,  $f_i$  is a  $(T_i, w_i)$ -partition of type  $B$  if and only if

$$0 < f_i(v_i) \leq f_i(u_i) \leq t.$$

Thus,

$$\Omega_{T_i}(w_i, t) = \binom{t+1}{2}.$$

Case 3:  $w_i(u_i) = 2i-1$  and  $w_i(v_i) = \overline{2i}$ . We see that  $f_i$  is a  $(T_i, w_i)$ -partition of type  $B$  if and only if

$$0 \leq f_i(v_i) < f_i(u_i) \leq t.$$

This implies that

$$\Omega_{T_i}(w_i, t) = \binom{t+1}{2}.$$

Case 4:  $w_i(u_i) = \overline{2i}$  and  $w_i(v_i) = \overline{2i-1}$ . In this case,  $f_i$  is a  $(T_i, w_i)$ -partition of type  $B$  if and only if

$$0 \leq f_i(v_i) \leq f_i(u_i) \leq t.$$

Hence,

$$\Omega_{T_i}(w_i, t) = \binom{t+2}{2}.$$

Combining the above four cases, we see that for any  $1 \leq i \leq n$ , the number of possible configurations of  $(T_i, w_i)$ -partitions of type  $B$  equals

$$3 \binom{t+1}{2} + \binom{t+2}{2} = (t+1)(2t+1).$$

It follows that

$$\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = ((t+1)(2t+1))^n,$$

as required. ■

### 3 Signed permutations and $I_{2n-1}$

In the previous section, we proved the conjecture of Savage and Visontai on the equidistribution of the descent number over signed permutations and the ascent number over  $s$ -inversion sequences in the set  $I_{2n}$ . In this section, we find a set  $V_n$  of signed permutations over which the descent number is equidistributed with the ascent number over

the set  $I_{2n-1}$ . Recall that  $I_{2n-1}$  is the set of  $s$ -inversion sequences of length  $2n - 1$  for  $s = (1, 4, 3, 8, 5, 12, \dots)$ . For an  $s$ -inversion sequence  $e = (e_1, e_2, \dots, e_n)$ , an ascent of  $e$  is defined as an integer  $i \in \{0, 1, \dots, n - 1\}$  such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

where we assume that  $e_0 = 0$  and  $s_0 = 1$ . The ascent number of  $e$  is meant to be the number of ascents of  $e$ . Define  $V_n(x)$  as the generating function of descent numbers of signed permutations in  $V_n$ . Recall that  $I_{2n-1}(x)$  denotes the generating function of ascent numbers of inversion sequences in  $I_{2n-1}$ . Savage and Schuster [4] showed that

$$\frac{I_{2n-1}(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n (2t+1)^{n-1} x^t. \quad (3.1)$$

We show that  $V_n(x)$  satisfies the same relation (3.1) as  $I_{2n-1}(x)$ . The proof is similar to that of Conjecture 1.1. So we reach the conclusion that  $V_n(x) = I_{2n-1}(x)$ .

Let  $U_n$  be the set of signed permutations on the multiset  $\{1^2, 2^2, \dots, (n-1)^2, n\}$ . Define  $V_n$  to be the subset of  $U_n$  consisting of signed permutations such that the element  $n$  carries a minus sign. Set

$$V_n(x) = \sum_{\sigma \in V_n} x^{\text{des}_B(\sigma)}.$$

We have the following equidistribution property.

**Theorem 3.1** *For  $n \geq 1$ , we have  $V_n(x) = I_{2n-1}(x)$ .*

*Proof.* In view of (3.1), we aim to show that

$$\frac{V_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n (2t+1)^{n-1} x^t. \quad (3.2)$$

Let  $F_n^*$  be the forest obtained from  $F_{n-1}$  by adding a single vertex  $v_n$  as a component. For example, Figure 3.2 illustrates the forest  $F_n^*$  for  $n = 3$ . Write  $L(F_n^*)$  for the set

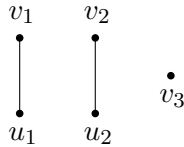


Figure 3.2: The forest  $F_n^*$  for  $n = 3$ .

of signed labelings  $w$  of  $F_n^*$  such that  $w(v_n) = -(2n - 1)$  and the labels on  $F_{n-1}$  form

a local signed labeling of  $F_{n-1}$ . Let  $Q_n(x)$  denote the generating function of descent numbers of linear extensions of  $F_n^*$  with signed labelings  $w \in L(F_n^*)$ , namely,

$$Q_n(x) = \sum_{w \in L(F_n^*)} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\text{des}_B(\sigma)}.$$

Analogous to the bijection  $\phi$  in the proof of Theorem 2.1, we can construct a descent preserving map  $\phi^*$  from the set

$$\bigcup_{w \in L(F_n^*)} \mathcal{L}(F_n^*, w)$$

to the set  $V_n$ . Let  $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n-1}$  be a linear extension in  $\mathcal{L}(F_n^*, w)$ , where  $w \in L(F_n^*)$ . Define  $\phi^*(\sigma) = \tau = \tau_1 \tau_2 \cdots \tau_{2n-1}$  as follows. For  $1 \leq i \leq 2n-1$ , assume that  $\tau_i$  has the same sign as  $\sigma_i$ . Then we set  $|\tau_i| = \frac{|\sigma_i|}{2}$  if  $|\sigma_i|$  is even and set  $|\tau_i| = \frac{|\sigma_i|+1}{2}$  if  $|\sigma_i|$  is odd. Clearly,  $\tau$  is a signed permutation in  $V_n$ . Moreover, one can construct the inverse of  $\phi^*$ , which is analogous to the inverse of  $\phi$ . This proves that  $\phi^*$  is a bijection. So we obtain that

$$V_n(x) = Q_n(x). \quad (3.3)$$

Thus (3.2) is equivalent to

$$\frac{Q_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n (2t+1)^{n-1} x^t. \quad (3.4)$$

By Theorem 2.4, the left-hand side of (3.4) can be written as

$$\frac{Q_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} \sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t) x^t.$$

Hence (3.4) is equivalent to

$$\sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t) = (t+1)^n (2t+1)^{n-1}. \quad (3.5)$$

The proof of (3.5) is similar to that for (2.7). For completeness, a detailed proof is presented. Recall that  $F_{n-1}$  contains  $n-1$  components  $T_1, T_2, \dots, T_{n-1}$ , where  $T_i$  is a tree rooted at  $v_i$  with  $u_i$  being the only child. Let  $T_n^*$  denote the component consisting of the single vertex  $v_n$ . By definition, the left-hand side of (3.5) equals the total number of  $(F_n^*, w)$ -partitions  $f$  of type  $B$  such that  $f(v) \leq t$  for any vertex  $v$  of  $F_n^*$ , where  $w$  is a signed labeling belonging to  $L(F_n^*)$ . Restricting  $f$  to  $F_{n-1}$ , we obtain a map  $f'$  from  $F_{n-1}$  to the set nonnegative integers. While, restricting  $f$  to  $T_n^*$ , we are led to a map  $f''$  from  $T_n^*$  to the set nonnegative integers. On the other hand, restricting  $w$  to  $F_{n-1}$  gives a local signed labeling  $w'$  of  $F_{n-1}$ , whereas restricting  $w$  to  $T_n^*$  gives a signed labeling  $w''$  of  $T_n^*$  such that  $w''(v_n) = -(2n-1)$ . Obviously,  $f'$  is a  $(F_{n-1}, w')$ -partition of type

$B$  satisfying the condition that  $f'(v) \leq t$  for any vertex  $v$  of  $F_{n-1}$ , and  $f''$  is a  $(T_n^*, w'')$ -partition of type  $B$  such that  $f''(v_n) \leq t$ . It can be seen that the above procedure is reversible. Hence we get

$$\sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t) = \Omega_{T_n^*}(w'', t) \sum_{w' \in L(F_{n-1})} \Omega_{F_{n-1}}(w', t). \quad (3.6)$$

In the proof of Theorem 2.2, it has been shown that

$$\sum_{w' \in L(F_{n-1})} \Omega_{F_{n-1}}(w', t) = ((t+1)(2t+1))^{n-1}. \quad (3.7)$$

To compute  $\Omega_{T_n^*}(w'', t)$ , we see that  $f''$  is a  $(T_n^*, w'')$ -partition of type  $B$  if and only if

$$0 \leq f''(v_n) \leq t.$$

Thus

$$\Omega_{T_n^*}(w'', t) = t + 1. \quad (3.8)$$

Combining (3.6), (3.7) and (3.8), we are led to (3.5). This completes the proof.  $\blacksquare$

## 4 Signed permutations and $I'_n$

In this section, we consider equidistributions of the descent number over signed permutations and the ascent number over  $s$ -inversion sequences for  $s = (2, 2, 6, 4, 10, 6, \dots)$ . Recall that the set of such  $s$ -inversion sequences of length  $n$  is denoted by  $I'_n$ . It turns out that we need to distinguish the parity of  $n$ .

First, we consider the case for  $I'_{2n}$ . Let  $I'_{2n}(x)$  be the generating function of ascent numbers of inversion sequences in  $I'_{2n}$ . Savage and Schuster [4] obtained a relation for the generating function of ascent numbers of  $s$ -inversion sequences for  $s = (1, 1, 3, 2, 5, 3, \dots)$ , that is,  $s_{2i} = i$  and  $s_{2i-1} = 2i - 1$  for  $i \geq 1$ . This leads to a relation satisfied by  $I'_{2n}(x)$ . As will be seen, this relation coincides with the relation (1.1) for  $I_{2n}(x)$ , and so we get  $I'_{2n}(x) = I_{2n}(x)$ . Since  $I_{2n}(x)$  equals the generating function  $P_n(x)$  of descent numbers of signed permutations on  $\{1^2, 2^2, \dots, n^2\}$ , we are led to the equidistribution as stated below.

**Theorem 4.1** *For  $n \geq 1$ , we have  $P_n(x) = I'_{2n}(x)$ .*

To prove the above theorem, we recall two formulas of Savage and Schuster [4] on the generating function of ascent numbers of  $s$ -inversion sequences of length  $n$ . For any sequence  $s = (s_1, s_2, \dots)$  of positive integers, let  $f_n^{(s)}(t)$  denote the number of sequences  $(a_1, a_2, \dots, a_n)$  of nonnegative integers such that

$$0 \leq \frac{a_1}{s_1} \leq \frac{a_2}{s_2} \leq \dots \leq \frac{a_n}{s_n} \leq t. \quad (4.1)$$

Savage and Schuster [4] deduced that

$$\frac{1}{(1-x)^{n+1}} \sum_e x^{\text{asc}(e)} = \sum_{t \geq 0} f_n^{(s)}(t) x^t, \quad (4.2)$$

where  $e$  ranges over  $s$ -inversion sequences of length  $n$ . For the sequence

$$s = (1, 1, 3, 2, 5, 3, \dots),$$

Savage and Schuster [4] showed that

$$f_n^{(s)}(t) = (t+1)^{\lfloor \frac{n}{2} \rfloor} \left( \frac{t+2}{2} \right)^{\lfloor \frac{n}{2} \rfloor}. \quad (4.3)$$

*Proof of Theorem 4.1.* Let

$$s = (2, 2, 6, 4, 10, 6, \dots)$$

and

$$s' = s/2 = (1, 1, 3, 2, 5, 3, \dots).$$

By (4.1), we see that

$$f_n^{(s)}(t) = f_n^{(s')}(2t).$$

Applying (4.3) to  $s'$ , we get

$$f_n^{(s)}(t) = (t+1)^{\lfloor \frac{n}{2} \rfloor} (2t+1)^{\lfloor \frac{n}{2} \rfloor}. \quad (4.4)$$

Let  $I'_n(x)$  be the generating function of ascent numbers of inversion sequences in  $I'_n$ . By (4.2) and (4.4), we obtain that

$$\frac{I'_n(x)}{(1-x)^{n+1}} = \sum_{t \geq 0} (t+1)^{\lfloor \frac{n}{2} \rfloor} (2t+1)^{\lfloor \frac{n}{2} \rfloor} x^t. \quad (4.5)$$

Replacing  $n$  with  $2n$  in (4.5), we arrive at

$$\frac{I'_{2n}(x)}{(1-x)^{2n+1}} = \sum_{t \geq 0} (t+1)^n (2t+1)^n x^t. \quad (4.6)$$

Comparing (4.6) with (1.1), we see that  $I'_{2n}(x)$  satisfies the same relation as  $I_{2n}(x)$ . This implies that  $I'_{2n}(x) = I_{2n}(x)$ . Since  $P_n(x) = I_{2n}(x)$ , we conclude that  $P_n(x) = I'_{2n}(x)$ . This completes the proof.  $\blacksquare$

We now consider the case for  $I'_{2n-1}$ . Recall that  $U_n$  is the set of signed permutations on  $\{1^2, 2^2, \dots, (n-1)^2, n\}$ . Let  $U_n(x)$  be the generating function of descent numbers of signed permutations in  $U_n$ . Replacing  $n$  with  $2n-1$  in (4.5), we find that

$$\frac{I'_{2n-1}(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^{n-1} (2t+1)^n x^t. \quad (4.7)$$

It will be shown that  $U_n(x)$  also satisfies relation (4.7). So we have the following equidistribution property for  $I'_{2n-1}$ .

**Theorem 4.2** For  $n \geq 1$ , we have  $U_n(x) = I'_{2n-1}(x)$ .

*Proof.* We proceed to show that

$$\frac{U_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^{n-1} (2t+1)^n x^t. \quad (4.8)$$

As defined in the proof of Theorem 3.1,  $F_n^*$  denotes the forest obtained from  $F_{n-1}$  by adding a single vertex  $v_n$  as a component  $T_n^*$ . We use  $L'(F_n^*)$  to stand for the set of signed labelings  $w$  of  $F_n^*$  such that  $w(v_n) = 2n-1$  or  $w(v_n) = -(2n-1)$ , and the restriction of  $w$  to  $F_{n-1}$  forms a local signed labeling of  $F_{n-1}$ . Let

$$H_n(x) = \sum_{w \in L'(F_n^*)} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\text{des}_B(\sigma)}. \quad (4.9)$$

Analogous to the construction of  $\phi^*$  in the proof of Theorem 3.1, we can establish a descent preserving bijection from the set

$$\bigcup_{w \in L'(F_n^*)} \mathcal{L}(F_n^*, w)$$

to the set  $U_n$ . This yields that

$$H_n(x) = U_n(x).$$

Therefore, (4.8) is equivalent to

$$\frac{H_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^{n-1} (2t+1)^n x^t. \quad (4.10)$$

By Theorem 2.4, for each signed labeling  $w \in L'(F_n^*)$ ,

$$\frac{1}{(1-x)^{2n}} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\text{des}_B(\sigma)} = \sum_{t \geq 0} \Omega_{F_n^*}(w, t) x^t.$$

It follows that

$$\frac{1}{(1-x)^{2n}} \sum_{w \in L'(F_n^*)} \sum_{\sigma \in \mathcal{L}(F_n^*, w)} x^{\text{des}_B(\sigma)} = \sum_{t \geq 0} \sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) x^t.$$

In view of the definition of  $G_n(x)$  as given in (4.9), we obtain that

$$\frac{H_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} \sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) x^t.$$

Hence (4.10) is equivalent to the following relation

$$\sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) = (t+1)^{n-1} (2t+1)^n. \quad (4.11)$$

The proof of (4.11) is similar to that of (3.5). Let  $w_1$  be the signed labeling of  $T_n^*$  such that  $w_1(v_n) = -(2n - 1)$ , and let  $w_2$  be the signed labeling of  $T_n^*$  such that  $w_2(v_n) = 2n - 1$ . Then

$$\sum_{w \in L(F_n^*)} \Omega_{F_n^*}(w, t) = (\Omega_{T_n^*}(w_1, t) + \Omega_{T_n^*}(w_2, t)) \sum_{w \in L(F_{n-1})} \Omega_{F_{n-1}}(w, t). \quad (4.12)$$

In the proof of Theorem 2.2, we have shown that

$$\sum_{w \in L(F_{n-1})} \Omega_{F_{n-1}}(w, t) = ((t + 1)(2t + 1))^{n-1}, \quad (4.13)$$

whereas in the proof of Theorem 3.1, we deduced that

$$\Omega_{T_n^*}(w_1, t) = t + 1. \quad (4.14)$$

Clearly, a map  $f$  from  $T_n^*$  to the set of nonnegative integers is a  $(T_n^*, w_2)$ -partition of type  $B$  if and only if  $0 < f(v_n) \leq t$ . Thus

$$\Omega_{T_n^*}(w_2, t) = t. \quad (4.15)$$

Substituting (4.13), (4.14) and (4.15) into (4.12), we arrive at (4.11). This completes the proof.  $\blacksquare$

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