The k-proper index of graphs*

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Abstract

A tree T in an edge-colored graph is a *proper tree* if any two adjacent edges of T are colored with different colors. Let G be a graph of order n and k be a fixed integer with $2 \leq k \leq n$. For a vertex set $S \subseteq V(G)$, a tree containing the vertices of S in G is called an *S-tree*. An edge-coloring of G is called a *k-proper coloring* if for every set S of k vertices in G, there exists a proper S-tree in G. The *k-proper index* of a nontrivial connected graph G, denoted by $px_k(G)$, is the smallest number of colors needed in a k-proper coloring of G. In this paper, we state some simple observations about $px_k(G)$ for a nontrivial connected graph G. Meanwhile, the k-proper indices of some special graphs are determined, and for every pair of positive integers a, b with $2 \leq a \leq b$, a connected graph G with $px_k(G) = a$ and $rx_k(G) = b$ is constructed for each integer k with $3 \leq k \leq n$. Also, we characterize the graphs with k-proper index $n - 1$ and $n - 2$, respectively.

Keywords: coloring of graphs, k-proper index, characterization of graphs

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1. Introduction

In this paper, all graphs under our consideration are finite, undirected, connected and simple. For more notation and terminology that will be used in the sequel, we refer to [2], unless otherwise stated.

In 2008, Chartrand et al. [8] first introduced the concept of rainbow connection. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \ldots, k\}(k \in \mathbb{N})$ is defined, where adjacent edges may be colored with the same color. For any two vertices u and v of G, a path in G

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connecting u and v is abbreviated as a uv-path. A uv-path P is a *rainbow uv-path* if no two edges of P are colored with the same color. The graph G is *rainbow connected* (with respect to c) if G contains a rainbow uv-path for every two vertices u and v, and the coloring c is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring*. The minimum k for which there exists a rainbow k-coloring of the edges of G is the *rainbow connection number* of G, denoted by $rc(G)$. The topic of rainbow connection is especially meaningful and numerous relevant papers have been written. For more details see a survey [23] and a book [24].

Subsequently, a series of generalizations of rainbow connection number were proposed. The k-rainbow index is one of them. An edge-colored tree T is a *rainbow tree* if no two edges of T are assigned the same color. Let G be a nontrivial connected graph of order n and let k be an integer with $2 \leq k \leq n$. A *k-rainbow coloring* of G is an edge coloring of G having the property that for every set S of k vertices of G, there exists a rainbow tree T in G such that $S \subseteq V(T)$. The minimum number of colors needed in a k-rainbow coloring of G is the *k-rainbow index* of G. These concepts were introduced by Chartrand et al. in [9], and were further studied in [4, 5, 10, 21, 22, 26].

In addition, a natural extension of the rainbow connection number is the proper connection number, which was introduced by Borozan et al. in [3]. A path in an edge-colored graph is said to be *properly edge-colored (or proper)*, if every two adjacent edges on the path differ in color. An edge-colored graph G is *k-proper connected* if any two vertices are connected by k internally pairwise vertex-disjoint proper paths. The *k-proper connection number* of a k-connected graph G, denoted by $pc_k(G)$, is defined as the smallest number of colors that are needed in order to make G k-proper connected. In particular, when $k = 1$, the 1-proper connection number is abbreviated as proper connection number and written as $pc(G)$. For more results, we refer to [1, 12, 13, 14, 15, 18, 25].

Inspired by the k-rainbow index and the proper connection number, a natural idea is to introduce the concept of k-proper index. A tree T in an edge-colored graph is a *proper tree* if any two adjacent edges of T are colored with different colors. Let G be a graph of order n and k be a fixed integer with $2 \leq k \leq n$. For a vertex set $S \subseteq V(G)$, a tree containing the vertices of S in G is called an *S-tree*. An edge-coloring of G is called a *k-proper coloring* if for every set S of k vertices in G, there exists a proper S-tree in G. The *k-proper index* of a nontrivial connected graph G, denoted by $px_k(G)$, is the smallest number of colors needed in a k-proper coloring of G. By definition,

$px_2(G)$ is precisely the proper connection number $pc(G)$ for any nontrivial graph G. As a variety of nice results about $pc(G)$ have been obtained, we in this paper only study $px_k(G)$ for $3 \leq k \leq n$.

The paper is organized as follows: In Section 2, some simple observations about $px_k(G)$ for a nontrivial graph G are stated. Meanwhile, certain necessary lemmas are also listed. In Section 3, the k-proper indices of some special graphs are determined. And for every pair of positive integers a, b with $2 \leq a \leq b$, a connected graph G with $px_k(G) = a$ and $rx_k(G) = b$ is constructed for each integer k with $3 \leq k \leq n$. In Section 4, the graphs with k-proper index $n - 1$ and $n - 2$ are characterized, respectively.

2. Preliminaries

We, in this section, state some observations about $px_k(G)$ for a nontrivial graph G. Also, certain necessary lemmas are listed.

For a graph G of order $n \geq 3$, it follows from the definition that

$$(*) \qquad pc(G) = px_2(G) \leq px_3(G) \leq px_4(G) \leq \cdots \leq px_n(G).$$

This simple property will be used frequently later.

Since any k-proper coloring of a spanning subgraph must be a k-proper coloring of its supergraph, then there exists a fundamental proposition about spanning subgraphs.

Proposition 1. *If G is a nontrivial connected graph of order $n \geq 3$ and H is a connected spanning subgraph of G, then $px_k(G) \leq px_k(H)$ for any k with $3 \leq k \leq n$. In particular, $px_k(G) \leq px_k(T)$ for every spanning tree T of G.*

It has been seen in [9] that $rx_k(G) \leq n - 1$ for any graph G of order $n \geq 3$ and any integer k with $3 \leq k \leq n$. Since a rainbow tree must be a proper tree, then obviously $px_k(G) \leq rx_k(G) \leq n - 1$. Moreover, this simple upper bound is sharp, the graphs with $px_k(G) = n - 1$ will be characterized in Section 4.

For any nontrivial graph G, $\chi'(G)$ denotes the edge-chromatic number of G. It is well-known that either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ by Vizing's Theorem, where $\Delta(G)$, or simply Δ, is the maximum degree of G. Accordingly, a natural upper bound on $px_k(G)$ with respect to these parameters follows.

Proposition 2. *Let G be a connected graph of order $n \geq 3$, the maximum degree $\Delta(G)$ and the edge-chromatic number $\chi'(G)$. Then for each integer k with $3 \leq k \leq n$, we have*

$$px_k(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

In addition, there is a classical result about the edge-chromatic number of a graph, which will be useful in the next section.

Lemma 1 ([2])**.** *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

For arbitrary k ($k \geq 3$) vertices of a nontrivial graph G, any tree T containing these vertices must contain internal vertices. While for any proper tree T, there must be $d(u)$ distinct colors assigned to the incident edges of each vertex u in T, where $d(u)$ denotes the degree of u in T. Hence, the incident edges of any internal vertex must be assigned with at least two distinct colors to make T proper. Then the following trivial lower bound is immediate.

Proposition 3. *For arbitrary connected graph G of order $n \geq 3$, we have*

$$px_k(G) \geq 2$$

for any integer k with $3 \leq k \leq n$.

Remark: The above lower bound of $px_k(G)$ is sharp since there exist many graphs satisfying $px_k(G) = 2$, as shown in Section 3. Further, we believe that it will be interesting to characterize all graphs with k-proper index 2 for specific values of k.

In any graph G, a path that contains every vertex of G is called a *Hamilton path* of G. A graph is *traceable* if it contains a Hamilton path. Bearing in mind that Inequality ($*$), the following is an immediate consequence of these definitions, as well as Proposition 3.

Proposition 4. *Let G be a connected graph of order $n \geq 3$, then $px_n(G) = 2$ if and only if G is a traceable graph.*

As mentioned before, characterizing all graphs with k-proper index 2 for specific values of k would be interesting. Proposition 4 is essentially the case of $k = n$. For the case of $k = n - 1$, there is a basic result that can be presented.

Observation 1. *If a connected graph G of order n satisfies $px_{n-1}(G) = 2$, then $px_n(G) = 2$ if and only if G is traceable. Otherwise, $px_n(G) = 3$.*

It is well known that if G is a simple graph of order $n \geq 3$ and minimum degree $\delta \geq \frac{n-1}{2}$, then G is traceable. Whereupon a direct corollary follows.

Corollary 1. *If G is a simple graph of order $n \geq 3$ and minimum degree $\delta \geq \frac{n-1}{2}$, then $px_k(G) = 2$ for each integer k with $3 \leq k \leq n$.*

In [9], Chartrand et al. derived the k-rainbow index of a nontrivial tree, which will be helpful in the next section.

Lemma 2 ([9])**.** *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$,*

$$rx_k(T) = n - 1.$$

In [3], Borozan et al. established the proper connection number of trees.

Lemma 3 ([3])**.** *If G is a tree then $pc(G) = \Delta(G)$.*

At the end of this section, we recall several notations required in the subsequent sections.

Let $E' \subseteq E(G)$ be a set of edges of a graph G, then $G[E']$ is the subgraph of G induced by E'. If e is an edge of G, then $G - e$ denotes the graph obtained from G by only deleting the edge e. If G is not complete, denote by $G + e$ the graph obtained from G by the addition of e, where e is an edge connecting two nonadjacent vertices of G.

3. The k-proper indices of special graphs

In this section, we determine the k-proper indices of complete graphs, cycles, wheels, trees and unicyclic graphs. Moreover, the independence of $px_k(G)$ and $rx_k(G)$ is given by a brief theorem.

By Proposition 4, if G is a traceable graph, then $px_k(G) = 2$. Obviously, the complete graphs, cycles and wheels are all traceable, thus the k-proper indices of these graphs are direct consequences of Proposition 4.

Theorem 1. *Let K_n, C_n and W_n be a complete graph, a cycle and a wheel with n ($n \geq 3$) vertices, respectively. Then for any integer k with $3 \leq k \leq n$, we have*

$$px_k(K_n) = px_k(C_n) = px_k(W_n) = 2.$$

Now we determine the k-proper index for a nontrivial tree.

Theorem 2. *If T is a tree of order $n \geq 3$, then for each integer k with $3 \leq k \leq n$,*

$$px_k(T) = \Delta(T).$$

Proof. Firstly, since T is bipartite, then $px_k(T) \leq \chi'(T) = \Delta(T)$ for $3 \leq k \leq n$ by Proposition 2 and Lemma 1. On the other hand, according to Inequality $(*)$ and Lemma 3, $px_k(T) \geq pc(T) = \Delta(T)$ holds naturally for $3 \leq k \leq n$. Therefore, we arrive at $px_k(T) = \Delta(T)$ for any integer k with $3 \leq k \leq n$. ∎

Combining with Proposition 1 and Theorem 2, one can check that the following proposition holds.

Proposition 5. *For any graph G of order $n \geq 3$ and any integer k with $3 \leq k \leq n$, we have*

$$px_k(G) \leq \min\{\Delta(T) : T \text{ is a spanning tree of } G\}.$$

Since $\Delta(T) \leq \Delta(G)$ for any spanning tree T of G, then the upper bound in Proposition 2 can be replaced by $\Delta(G)$.

Proposition 6. *Let G be a connected graph of order $n \geq 3$ and maximum degree $\Delta(G)$, then*

$$px_k(G) \leq \Delta(G)$$

for each integer k with $3 \leq k \leq n$.

Remark: The above upper bound of $px_k(G)$ is sharp since the equality clearly holds for any nontrivial tree.

In order to get the k-proper index of a unicyclic graph, an assistant lemma is presented.

Lemma 4. *Let G be a connected graph of order $n \geq 3$ containing bridges and v be any vertex of G. Denote by $b(v)$ the number of bridges incident with v. Set $b = \max\{b(v) : v \in V(G)\}$. Then for each integer k with $3 \leq k \leq n$, we have $px_k(G) \geq b$.*

Proof. Since for $3 \leq k \leq n$, it has been seen from Inequality $(*)$ that $px_k(G) \geq px_3(G)$, then we should only prove the case when $k = 3$. Since $px_3(G) \geq 2$ by Proposition 3, the result is trivial when $b = 1$ or 2. Thus we may assume that $b \geq 3$. Let u be a vertex with $b(u) = b = \max\{b(v) : v \in V(G)\}$. Let $F = \{uw_1, uw_2, \ldots, uw_b\}$ be the set of bridges incident with u. Set $A = \{u, w_1, w_2, \ldots, w_b\}$. For any 3-set $S = \{w_i, w_j, u\} \subseteq A$, where $i, j \in \{1, 2, \ldots, b\}$ and $i \neq j$, every S-tree T must contain the edges uw_i and uw_j. Hence, the edges uw_i and uw_j receive distinct colors to make T proper, which implies that the edges $uw_1, uw_2, \ldots, uw_b$ need b distinct colors in any 3-proper coloring of G. Therefore, $px_3(G) \geq b$. This completes the proof. ∎

With the aid of Lemma 4, now we are able to deal with the k-proper index for a unicyclic graph.

Theorem 3. *Let G be a unicyclic graph of order $n \geq 3$, and maximum degree $\Delta(G)$. Then, for each integer k with $3 \leq k \leq n$,*

$$px_k(G) = \Delta(G) - 1$$

when G contains at most two vertices having maximum degree such that the vertices with maximum degree are all in the unique cycle of G and these two vertices (if both exist) are adjacent;
Otherwise,

$$px_k(G) = \Delta(G).$$

Proof. Note that when $G = C_n$, it follows from Theorem 1 that $px_k(G) = px_k(C_n) = 2 = \Delta(G)$ for $3 \leq k \leq n$. Thus in the following we assume that G is not a cycle. And assume the vertices in the unique cycle of G are $u_1, u_2, \ldots, u_g$. Also keep in mind that $px_k(G) \leq \Delta(G)$ for $3 \leq k \leq n$, which will be used later. As before, denote by $b(v)$ the number of bridges incident with the vertex v. The discussion is divided into three cases.

Case 1. At first, assume that G contains a vertex, say u, satisfying
(1) the degree of u is $d(u) = \Delta(G)$.
(2) u is not in the cycle of G.

Then the incident edges of u are all bridges, i.e., $b(u) = d(u) = \Delta(G)$. According to Lemma 4, we directly have $px_k(G) \geq b(u) = \Delta(G)$ for $3 \leq k \leq n$. Meanwhile, Proposition 6 guarantees $px_k(G) \leq \Delta(G)$ for $3 \leq k \leq n$. Accordingly, we get $px_k(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$ in this case.

By Case 1, if such a vertex u exists in G, then we always have $px_k(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$. To avoid redundant presentation, we in the following suppose that G contains no such vertices.

Case 2. Now assume G simultaneously satisfies
(3) G contains at most two vertices having maximum degree;

(4) the vertices with maximum degree are all in the unique cycle of G;
(5) these two vertices (if both exist) are adjacent in G.

Then without loss of generality, suppose that $d(u_1) = \Delta(G)$, $d(u_2) \leq \Delta(G)$ and $d(u) < \Delta(G)$ for any other vertex u. Moreover, suppose that the neighbors of u_1 are v_1, v_2, $\ldots$, $v_{\Delta(G)-2}$, $v_{\Delta(G)-1} = u_2$ and $v_{\Delta(G)} = u_g$. Thereupon, in any 3-proper coloring c of G, based on the proof of Lemma 4, the edges u_1v_i with $i \in \{1, 2, \ldots, \Delta(G) - 2\}$ are assigned with $\Delta(G) - 2$ distinct colors since they are all bridges incident with u_1. Without loss of generality, suppose that $c(u_1v_1) = 1$, $c(u_1v_2) = 2$, $\ldots$, $c(u_1v_{\Delta(G)-2}) = \Delta(G) - 2$. Further, we claim that at least one new color is used by the edges u_1u_2 and u_1u_g. For otherwise, suppose that $c(u_1u_2) = i$ and $c(u_1u_g) = j$ with $i, j \in \{1, 2, \ldots, \Delta(G) - 2\}$. If $i = j$, then there exists no proper tree containing the vertices u_1, u_2 and v_i, a contradiction. If $i \neq j$, then there exists no proper tree containing the vertices v_i, v_j and u_2, again a contradiction. Therefore, at least $\Delta(G)-2+1 = \Delta(G)-1$ different colors are used by c. It follows that $px_3(G) \geq \Delta(G) - 1$. Thus, Inequality $(*)$ deduces that $px_k(G) \geq px_3(G) \geq \Delta(G)-1$ for each integer k with $3 \leq k \leq n$. On the other hand, obviously $G - u_1u_2$ is a spanning tree of G with maximum degree $\Delta(G) - 1$. By Theorem 2, we know that $px_k(G - u_1u_2) = \Delta(G - u_1u_2) = \Delta(G) - 1$ for $3 \leq k \leq n$. Consequently, $px_k(G) \leq px_k(G - u_1u_2) = \Delta(G) - 1$ based on Proposition 1. To sum up, we obtain $px_k(G) = \Delta(G) - 1$ for each integer k with $3 \leq k \leq n$ in this case.

Case 3. Finally, we discuss the case when G contains at least two vertices u_i and u_j such that
(6) $d(u_i) = d(u_j) = \Delta(G)$;
(7) both u_i and u_j are in the cycle of G;
(8) u_i and u_j are not adjacent in G.

Then we claim that $px_3(G) \geq \Delta(G)$. Assume to the contrary, $px_3(G) \leq \Delta(G)-1$. Let c' be a 3-proper coloring of G using colors from $\{1, 2, \ldots, \Delta(G)-1\}$. Let the neighbors of u_i be w_1, w_2, $\ldots$, $w_{\Delta(G)-2}$, $w_{\Delta(G)-1} = u_{i-1}$, $w_{\Delta(G)} = u_{i+1}$, and the neighbors of u_j be z_1, z_2, $\ldots$, $z_{\Delta(G)-2}$, $z_{\Delta(G)-1} = u_{j-1}$, $z_{\Delta(G)} = u_{j+1}$. Similarly to Case 2, the edges u_iw_t with $t \in \{1, 2, \ldots, \Delta(G) - 2\}$ are assigned with $\Delta(G) - 2$ distinct colors. Without loss of generality, suppose that $c'(u_iw_1) = 1$, $c'(u_iw_2) = 2$, $\ldots$, $c'(u_iw_{\Delta(G)-2}) = \Delta(G) - 2$. Thus, either $c'(u_iu_{i-1}) = c'(u_iu_{i+1}) = \Delta(G) - 1$, or there exists at least one edge between u_iu_{i-1} and u_iu_{i+1}, say u_iu_{i-1}, such that $c'(u_iu_{i-1}) = x_1$ with $x_1 \in \{1, 2, \ldots, \Delta(G) - 2\}$. Similarly, the edges u_jz_t with $t \in \{1, 2, \ldots, \Delta(G) - 2\}$ also receive $\Delta(G) - 2$ distinct colors. And for the edges u_ju_{j-1} and u_ju_{j+1},

either $c'(u_ju_{j-1}) = c'(u_ju_{j+1})$, or there exists at least one of them, say u_ju_{j+1}, such that $c'(u_ju_{j+1}) = c'(u_jz_{x_2})$ with $x_2 \in \{1, 2, \ldots, \Delta(G) - 2\}$.
(i) If $c'(u_iu_{i-1}) = c'(u_iu_{i+1})$ and $c'(u_ju_{j-1}) = c'(u_ju_{j+1})$, then there exists no proper tree containing the vertices u_{i-1}, u_{i+1} and w_1, a contradiction.
(ii) If $c'(u_iu_{i-1}) = c'(u_iu_{i+1})$ and $c'(u_ju_{j+1}) = c'(u_jz_{x_2})$ with $x_2 \in \{1, 2, \ldots, \Delta(G)-2\}$, then there exists no proper tree containing the vertices u_{j+1}, u_j and z_{x_2}, a contradiction.
(iii) If $c'(u_iu_{i-1}) = x_1$ with $x_1 \in \{1, 2, \ldots, \Delta(G) - 2\}$ and $c'(u_ju_{j-1}) = c'(u_ju_{j+1})$, then there exists no proper tree containing the vertices u_{i-1}, u_i and w_{x_1}, a contradiction.
(iv) If $c'(u_iu_{i-1}) = x_1$ with $x_1 \in \{1, 2, \ldots, \Delta(G) - 2\}$ and $c'(u_ju_{j+1}) = c'(u_jz_{x_2})$ with $x_2 \in \{1, 2, \ldots, \Delta(G) - 2\}$, then there exists no proper tree containing the vertices w_{x_1}, u_{i-1} and z_{x_2}, a contradiction.

In summary, we verify that $px_3(G) \geq \Delta(G)$, which deduces that $px_k(G) \geq px_3(G) \geq \Delta(G)$ for $3 \leq k \leq n$. Combining with $px_k(G) \leq \Delta(G)$ for $3 \leq k \leq n$, we at last arrive at $px_k(G) = \Delta(G)$ for each integer k with $3 \leq k \leq n$ in this case.

The proof of this theorem is finished. ∎

We conclude this section with a simple theorem to address the independence of $px_k(G)$ and $rx_k(G)$. Notice that $px_k(G) \leq rx_k(G)$ always holds, thus we can construct a connected graph G with $pxk(G) = a$ and $rx_k(G) = b$, where $a \leq b$.

Theorem 4. *For every pair of positive integers a, b with $2 \leq a \leq b$, there exists a connected graph G such that $px_k(G) = a$ and $rx_k(G) = b$ for each integer k with $3 \leq k \leq n$.*

Proof. For each pair of positive integers a, b with $2 \leq a \leq b$, let G be a nontrivial tree of order $n = b + 1$ and maximum degree $\Delta(G) = a$. The existence of such a tree is guaranteed by $2 \leq a \leq b$. Then based on Theorem 2 and Lemma 2, we know that $px_k(G) = \Delta(G) = a$ and $rx_k(G) = n - 1 = b$ for each integer k with $3 \leq k \leq n$. The proof is thus complete. ∎

4. Graphs with k-proper index $n - 1$, $n - 2$

In this section, we are going to characterize the graphs whose k-proper index equals to $n - 1$ and $n - 2$, respectively, where $3 \leq k \leq n$. First of all, we give the following lemma that will be used in the sequel.

Lemma 5. *For $n \geq 5$, let S_n^+ be the graph of order n obtained by adding a new edge to the n-vertices star S_n, and S_n^{++} be the graph of order n obtained by adding a new edge to S_n^+. Then we have $px_k(S_n^{++}) \leq n-3$ for each integer k with $3 \leq k \leq n$.*

Proof. Let $V(S_n^+) = V(S_n^{++}) = \{u, v_1, v_2, \ldots, v_{n-1}\}$. Without loss of generality, set $d_{S_n^+}(u) = d_{S_n^{++}}(u) = n-1$ and $d_{S_n^+}(v_1) = d_{S_n^+}(v_2) = 2$. Further, let e be the edge of S_n^{++} added to S_n^+. We split the remaining proof into the following two cases depending on the position of e.

Case 1. The edges e and v_1v_2 are vertex-disjoint. Without loss of generality, let $e = v_3v_4$. Then, $G' = G - uv_1 - uv_3$ is a spanning tree of S_n^{++} with maximum degree $n-1-2 = n-3$. It follows from Theorem 2 that $px_k(G') = \Delta(G') = n-3$ for $3 \leq k \leq n$. Hence, Proposition 1 deduces that $px_k(S_n^{++}) \leq px_k(G') = n-3$ for each integer k with $3 \leq k \leq n$.

Case 2. The edges e and v_1v_2 have a common vertex. Without loss of generality, let $e = v_2v_3$. Then, $G'' = G - uv_2 - uv_3$ is a spanning tree of S_n^{++} with maximum degree $n-1-2 = n-3$. Similarly, $px_k(G'') = \Delta(G'') = n-3$ for $3 \leq k \leq n$. Hence, we can also get $px_k(S_n^{++}) \leq px_k(G'') = n-3$ for each integer k with $3 \leq k \leq n$.

Combining the above two cases, now the lemma follows. ∎

Theorem 5. *Let G be a connected graph of order n $(n \geq 4)$. Then for each integer k with $3 \leq k \leq n$, we have $px_k(G) = n-1$ if and only if $G \cong S_n$, where S_n is the star of order n.*

Proof. Firstly, if $G \cong S_n$, then by Theorem 2, we directly obtain $px_k(G) = px_k(S_n) = \Delta(S_n) = n-1$ for $3 \leq k \leq n$.

Conversely, suppose G is a connected graph with $px_k(G) = n-1$ for each integer k with $3 \leq k \leq n$. Since $n-1 = px_k(G) \leq \Delta(G)$ by Proposition 6, meanwhile $\Delta(G) \leq n-1$ holds for any simple graph of order n. Then, $\Delta(G) = n-1$. The hypothesis is true if $G \cong S_n$. If $G \ncong S_n$, let u be a vertex of G with $d(u) = \Delta(G) = n-1$. Let $V(G)\backslash u = \{v_1, v_2, \ldots, v_{n-1}\}$ denote the set of the remaining vertices in G. Since $G \ncong S_n$, there exist at least two vertices, say v_1 and v_2, such that they are adjacent in G. Set $G' = G[\bigcup_{i=1}^{n-1} uv_i] + v_1v_2$. Then, as $n \geq 4$, G' is a unicyclic graph satisfying the conditions in Case 2 of Theorem 3 with maximum degree $n-1$. Hence, $px_k(G') = \Delta(G') - 1 = n-2$ by Theorem 3. Certainly, G' is a spanning subgraph of G, therefore $px_k(G) \leq$

$px_k(G') = n-2$ for $3 \le k \le n$ according to Proposition 1, contradicting our assumption that $px_k(G) = n-1$. Consequently, $G \cong S_n$.

The proof is thus complete. ∎

Remark: If G is a connected graph of order $n = 3$, then one can check that $px_3(G) = n-1 = 2$ if and only if $G \cong S_3$ or $G \cong C_3$.

Theorem 6. *Let G be a connected graph of order n ($n \ge 5$). Then for each integer k with $3 \le k \le n$, we have $px_k(G) = n-2$ if and only if $G \cong S_n^+$ or G_0, where S_n^+ is defined in Lemma 5 and G_0 is shown in Figure 1.*

Proof. On one hand, if $G \cong S_n^+$, then G is a unicyclic graph with maximum degree $n-1$ satisfying the conditions in Case 2 of Theorem 3. Thus $px_k(G) = px_k(S_n^+) = \Delta(G) - 1 = n-2$ for $3 \le k \le n$. If $G \cong G_0$, then G is a tree of order $n \ge 5$ and maximum degree $n-2$. Accordingly, by Theorem 2, $px_k(G) = px_k(G_0) = \Delta(G) = n-2$ for $3 \le k \le n$.

On the other hand, if $px_k(G) = n-2$, then by Proposition 6, $\Delta(G) \ge px_k(G) = n-2$, which means that $\Delta(G) = n-2$ or $n-1$. The remaining proof is divided into two cases depending on the value of $\Delta(G)$.

Case 1. $\Delta(G) = n-1$.

In this case, since $px_k(S_n) = n-1$ for $3 \le k \le n$, as shown before, then G must contain S_n^+ as a connected spanning subgraph. If $G \cong S_n^+$, we have known that $px_k(S_n^+) = n-2$ for $3 \le k \le n$. Now suppose $G \ncong S_n^+$. Then there exists a connected spanning subgraph with the form of S_n^{++} in G. Applying Proposition 1 together with Lemma 5, we arrive at $px_k(G) \le px_k(S_n^{++}) \le n-3$, a contradiction. Hence, $G \cong S_n^+$ in this case.

Case 2. $\Delta(G) = n-2$.

Then G_0 must be a connected spanning subgraph of G. If $G \cong G_0$, then $px_k(G_0) = n-2$ for $3 \le k \le n$. If $G \ncong G_0$, then there exists at least one edge $e \in E(G) \setminus E(G_0)$. Thus, G contains a connected spanning subgraph isomorphic to G_1, G_2 or G_3, where G_1, G_2 and G_3 are shown in Figure 1. Clearly, one can check that G_1, G_2 and G_3 are all unicyclic graphs with maximum degree $n-2$ satisfying the conditions in Case 2 of Theorem 3. Thereupon, by Theorem 3 as well as Proposition 1, we directly get that $px_k(G) \le px_k(G_i) = \Delta(G_i) - 1 = n-3$ for $3 \le k \le n$ and $i = 1, 2$ or 3, which is a contradiction. Accordingly, $G \cong G_0$ in this case.

In summary, if $px_k(G) = n-2$ for $3 \le k \le n$, then $G \cong S_n^+$ or $G \cong G_0$. And the proof of this theorem is complete. ∎

Remark: When $n = 4$, except for the star S_4, other connected graphs of order 4 are all traceable. Then by Proposition 4, the k-proper indices of these graphs equal to $2 = n - 2$ for each integer k with $3 \leq k \leq 4$. While for the star S_4, we know that $px_k(S_4) = 3$ for $3 \leq k \leq 4$. Consequently, we can easily claim that if G is a connected graph of order $n = 4$, then for each integer k with $3 \leq k \leq 4$, $px_k(G) = n - 2 = 2$ if and only if $G \ncong S_4$.

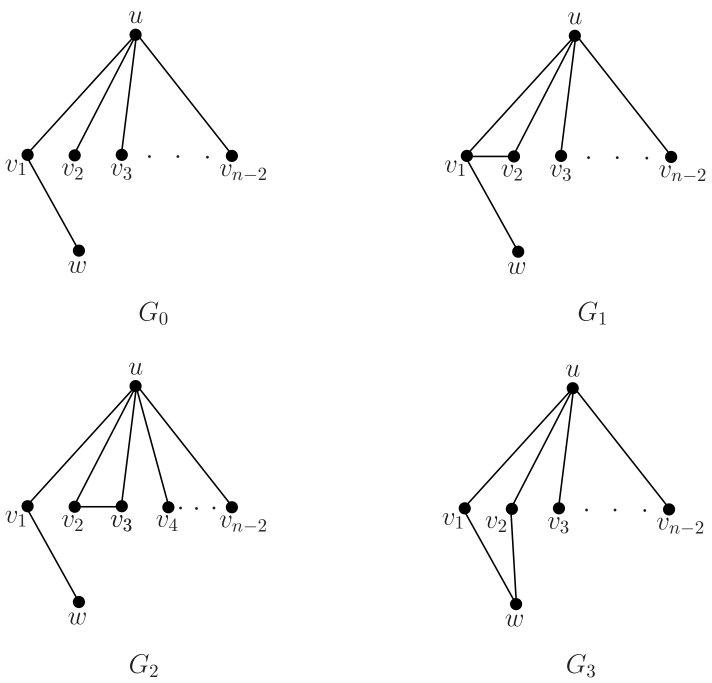

Figure 1: The graphs G_i for $i = 0, 1, 2, 3$.

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