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Light Edges in 3-Connected 2-Planar Graphs With Prescribed Minimum Degree

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Abstract A graph is called 2-*planar* if it can be drawn in the plane such that each edge is crossed by at most other two edges. The *weight* of an edge is the sum of degrees of its ends. In the present paper, we focus on 3-connected 2-planar graphs with minimum degree 6 and show the existence of edges with weight at most 30 by a discharging process.

Keywords 2-planar graph · Light edge · Weight

Mathematics Subject Classification 05C10 · 68R10

1 Introduction

All graphs considered in this paper are finite, simple, undirected, and connected. We follow [1] for the notation and terminology not defined here.

Let G be a graph. We denote by $V(G)$, $E(G)$, and $\delta(G)$ the vertex set, edge set, and minimum degree of G, respectively. A vertex of G is called a k-*vertex* if it has degree k in G. The *weight* of an edge in G is defined as the sum of degrees of its ends. An edge of G is called a *light edge* if it has the minimum weight. (In some earlier papers,

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"light edge" is defined as an edge with weight 13. But in [9], the meaning of "light edge" is changed, and in the present paper, we use the definition in [9].)

The interest of light edges stemmed from a result of Kotzig [11], which says that every 3-dimensional polyhedral graph (i.e., 3-connected planar graph) contains an edge with weight at most 13, and at most 11 in the absence of 3-vertex. These bounds are sharp and one can see that by some appropriate iteration of the icosahedron and the dodecahedron. On basis of the work of Grünbaum [8], Erdős conjectured that Kotzig's conclusion holds for every planar graph with minimum degree at least 3. This conjecture was proved by Barnette (unpublished, see [8]) and by Borodin [3] in 1989 independently. For more results in this topic, the reader may refer [10].

Let G be a graph. A drawing of G means a representation of it on the plane such that (1) the vertices are represented by distinct points of the plane; (2) every edge is represented by a Jordan arc connecting the ends of this edge but not passing through any other vertex; and (3) any two edges have finite crossings in common, and any three edges have not crossings in common. Let k be a nonnegative integer. A drawing of G is called *k-planar* if each edge is crossed by at most k other edges, and G is a *k-planar* graph if it admits a k-planar drawing.

Interest in k-planar graphs stems from the work on a coloring problem of Ringel [12], who considered a simultaneous vertex-face coloring of plane graphs and conjectured that, for this type of coloring, 6 colors suffice (note that this coloring corresponds to a regular coloring of underlying vertex-face adjacency/incidence graph which is 1-planar). Ringel's Conjecture was proved by Borodin in [2,3] through different approaches. Since then, the study on k-planar graphs has received considerable attention in the literature (see, for example, [4,6,7,13–16]).

In 2007, Fabrici and Madaras [7] showed that each light edge in a 3-connected 1-planar graph has weight at most 40. As observed in [7], the bound 40 may not be the best. For a 1-planar graph G with $\delta(G) \geq 4$, Hudák and Šugerek [9] proved that every light edge in G has weight no more than 17, and in particular, each light edge has weight 14 if further $\delta(G) > 4$.

In this paper, we focus on the light edges of 2-planar graphs and prove the following result.

Theorem 1.1 *If G is a 3-connected 2-planar graph with $\delta(G) \geq 6$, then there is an edge of G with ends of degree at most 15; in particular, each light edge of G has weight at most 30.*

2 Proof of Theorem 1.1

Suppose that there are counterexamples to Theorem 1.1. Choose a counterexample G on a given number, say n, of vertices such that G has maximum number of edges. Let D be an optimal 2-planar drawing of G, that is, D has the minimum number of crossings. Construct a plane graph $D^\times$ from D by identifying every crossing with a new 4-vertex. In the resulting graph $D^\times$, those new 4-vertices are called *false vertices* and the other vertices are called *true vertices*.

By [13, Lemma 1.1], we always assume that, for an optimal 2-planar drawing, every pair of edges has at most one point in common, where "one point" may be a vertex or a crossing. Thus $D^{\times}$ is a simple graph. Moreover, it is easy to show that $D^{\times}$ is 2-connected, and so each face has a cycle of $D^{\times}$ as boundary.

Let V and F be the vertex set and face set of $D^{\times}$, respectively. For $v \in V$ and $f \in F$, denote by $\deg(v)$ and $\deg(f)$ the degree of v and the size of f in $D^{\times}$, respectively. A face $f \in F$ is called a d-face if $\deg(f) = d$.

For every d-vertex $v \in V = V(D^{\times})$, the edges in $D^{\times}$ incident with v form a d-tuple in the anticlockwise order around v, which results a d-tuple, denoted by $T(v)$, of the neighbors of v.

Since G is a counterexample to Theorem 1.1, we know that for every $uu' \in E(G)$, one of u and u' must have degree at least 16. For convenience, we call a vertex $v \in V$ a *big vertex* if $\deg(u) \geq 16$, and a *small vertex* otherwise.

Denote by W the set of false vertices in $D^{\times}$.

Lemma 2.1 *Let u be a big vertex and $T(u) = (v_1, v_2, \ldots, v_d)$, where $d = \deg(u)$. Suppose that there are $1 \leq i \leq d$ and $0 \leq r \leq d-1$ such that $v_i, v_{i+1}, \ldots, v_{i+r} \in W$, where the subscripts take modulo d. Then $r = 0$ or 1.*

Proof Without loss of generality, we assume that $v_1, \ldots, v_{1+r} \in W$ for some $1 \leq r \leq d-1$. We shall show $r = 1$. Consider the drawing D of G.

Take two edges uu_1 and $u'u'_1$ of G which cross each other in D at v_1. Since D is a 2-planar drawing, we may assume that $u'v_1 \in E(D^{\times})$. Suppose that there is no edge in G joins u and u'. Then we may get a 2-planar drawing of some graph G_1 from D by adding a suitable Jordan arc connecting the points u and u'. Note that u is a big vertex. Then we get a counterexample G_1 to Theorem 1.1; however, $|E(G_1)| = |E(G)| + 1$, which contradicts the choice of G. Therefore, $uu' \in E(G)$; in particular, u' is a neighbor of u in $D^{\times}$.

Recalling that D is an optimal 2-planar drawing of G, we conclude that uu' contains no crossings. Then $uu'v_1u$ is a 3-cycle of $D^{\times}$. Assume that u_1 lies outside the 3-cycle $uu'v_1u$. If the interior of $uu'v_1u$ contains some vertices of $D^{\times}$, then they must contain true vertices, and so we get a 2-vertex-cut $\{u, u'\}$ of G, a contradiction. Then we have a face f_1 (of $D^{\times}$) with boundary $uu'v_1u$. By the definition of $T(u)$, we have $u' = v_d$ as $v_2 \in W$ and u' is a true vertex.

Let f_2 be the other face of $D^{\times}$ incident with uv_1. Then f_2 is incident with v_2. Let $k = \deg(f_2)$. Since G is 3-connected, $D^{\times}$ is 2-connected. Thus the boundary of every face of $D^{\times}$ is a cycle. Assume that the boundary of f_2 is a k-cycle $x_1, x_2, x_3, \ldots, x_{k-1}, x_kx_1$, where $x_1 = u$, $x_2 = v_1$, and $x_k = v_2$. Without loss of generality, we assume that f_2 is a bounded face. Suppose that x_{k-1} is a true vertex (so $k \geq 4$). Then we claim that $ux_{k-1} \in E(G)$. If not, then we may get a 2-planar drawing of some graph G_2 from D by adding a Jordan arc in the interior of f_2 connecting the points u and x_{k-1}, thus $ux_{k-1} \in E(G)$. Since f_2 is a face, ux_{k-1} is located outside f_2. Moreover, ux_{k-1} has no crossing; otherwise, we can redraw ux_{k-1} in the interior of f_2 to loss this crossing. Since f_2 is a face, uv_2 and v_2x_{k-1} have no crossing. Thus $uv_2x_{k-1}u$ is a cycle of $D^{\times}$. Note that there are some true vertices in the two sides of $uv_2x_{k-1}u$. That means $\{u, x_{k-1}\}$ is a 2-vertex-cut of G, which contradicts the 3-connectivity of G. Therefore, x_{k-1} is a false vertex. Then there is an edge $u''u''_1$ of G such that the edge x_kx_{k-1} of $D^{\times}$ is contained in $u''u''_1$ in the drawing D. Assume that u'', x_k, x_{k-1}, and u''_1 lie on edge $u''u''_1$ in succession. Note that D is a 2-planar drawing. Then $u''x_k$ is an edge of $D^\times$. Since $uv_2,\ u''v_2 \in E(D^\times)$, there is no crossing lying inside uv_2 and $u''v_2$, respectively.

Suppose that there is no edge in G joins u and u''. Then we may get a 2-planar drawing of some graph G_3 from D by adding a suitable Jordan arc connecting the points u and u''. Note that u is a big vertex. Then we get a counterexample G_3 to Theorem 1.1; however, $|E(G_3)| = |E(G)| + 1$, which contradicts the choice of G. Therefore, $uu'' \in E(G)$; in particular, u'' is a neighbor of u in $D^\times$.

Recalling that D is an optimal 2-planar drawing of G, we conclude that uu'' contains no crossings. Then $uu''v_2u$ is a 3-cycle of $D^\times$. Assume that u_1'' lies outside $uu''v_2u$. If the interior of $uu''v_2u$ contains some vertices of $D^\times$, then they must contain true vertices, and so we get a 2-vertex-cut $\{u, u''\}$ of G, a contradiction. Then we have a face f_3 (of $D^\times$) with boundary $uu''v_2u$. By the definition of $T(u)$, we have $u'' = v_3$. Hence $r = 1$. □

For a true vertex u, denote by $\deg_t(u)$ the number of true neighbors of u in $D^\times$. Then, by Lemma 2.1, the following corollary holds.

Corollary 2.2 *If u is a big vertex, then* $\deg_t(u) \geq \left\lceil \frac{\deg(u)}{3} \right\rceil \geq 6$.

We shall use a discharging method on $D^\times$ to deduce a contradiction. Assign the initial charge by

$$c(x) = \begin{cases} \deg(x) - 6, & \text{if } \ x \in V = V(D^\times); \\ 2\deg(x) - 6, & \text{if } \ x \in F = F(D^\times). \end{cases}$$

Then we get the following equation according to Euler polyhedral formula,

$$\sum_{x \in V \cup F} c(x) = \sum_{v \in V} (\deg(v) - 6) + \sum_{f \in F} (2\deg(f) - 6) = -12 < 0. \tag{1}$$

Next we redistribute the charge values $c(x)$, $x \in V \cup F$ by two rules such that the total charge sum remains the same. For a face $f \in F$, denote by $\deg_t(f)$ the number of true vertices incident with f.

Rule 1 *Every true vertex u with* $\deg(u) > \deg_t(u)$ *sends* $\frac{\deg(u)-6}{\deg(u)-\deg_t(u)}$ *to every false neighbor.*
Rule 2 *Every face f with* $\deg(f) > \deg_t(f)$ *sends* $\frac{2\deg(f)-6}{\deg(f)-\deg_t(f)}$ *to every incident false vertex.*

Denote by c' the resulting charge after the application of Rules 1 and 2. Let W be the set of false vertices of $D^\times$. Then for $x \in (V \setminus W) \cup F$, either $c'(x) = 0$ or $\deg(x) = \deg_t(x)$ and $c'(x) = c(x)$. Thus

$$\sum_{w \in W} c'(w) \leq \sum_{x \in V \cup F} c'(x) = \sum_{x \in V \cup F} c(x) < 0. \tag{2}$$

Next we shall deduce a contradiction by proving

$$\sum_{w \in W} c'(w) \geq 0.$$

Consider the subgraph $D^{\times}[W]$ of $D^{\times}$ induced by W. Since D is a 2-planar drawing, we know that every vertex of $D^{\times}[W]$ has degree at most 2. Recalling that $D^{\times}$ is simple, every component of $D^{\times}[W]$ is either a path or a cycle of length at least three.

Lemma 2.3 *Let H be a component of $D^{\times}[W]$. If H is a cycle, then*

$$\sum_{w \in V(H)} c(w) \geq 0.$$

Proof Note that all vertices of H are false. Since D is an optimal 2-planar drawing of G, we conclude that H, as a cycle of $D^{\times}$, is the boundary of a face f of $D^{\times}$.

Let $H = w_1 w_2 \cdots w_s w_1$, where $s \geq 3$ (since $D^{\times}$ is simple, by [13, Lemma 1.1]). Take edges $u_i u'_{i+1} \in E(G)$ such that $u_i u'_{i+1}$ crosses $u_{i-1} u'_i$ and $u_{i+1} u'_{i+2}$ at w_i and w_{i+1}, respectively, where the subscripts take modulo s. Denote by $f_{\{i,i+1\}}$ the face of $D^{\times}$ other than f which is incident with $w_i w_{i+1}$, reading the subscripts modulo s. Without loss of generality, we assume that $f_{\{i,i+1\}}$ is incident with u'_i and u_{i+1}. Let f_i be the face of $D^{\times}$ incident with u_i, u'_i, and w_i, see Fig. 1. (Note that some vertices may be identical.) Then

$$\begin{aligned} c'(w_i) = c(w_i) &+ \frac{\deg(u_i) - 6}{\deg(u_i) - \deg_t(u_i)} + \frac{\deg(u'_i) - 6}{\deg(u'_i) - \deg_t(u'_i)} \\ &+ \frac{2\deg(f) - 6}{\deg(f) - \deg_t(f)} + \frac{2\deg(f_i) - 6}{\deg(f_i) - \deg_t(f_i)} \\ &+ \frac{2\deg(f_{\{i,i+1\}}) - 6}{\deg(f_{\{i,i+1\}}) - \deg_t(f_{\{i,i+1\}})} + \frac{2\deg(f_{\{i-1,i\}}) - 6}{\deg(f_{\{i-1,i\}}) - \deg_t(f_{\{i-1,i\}})} \end{aligned}$$

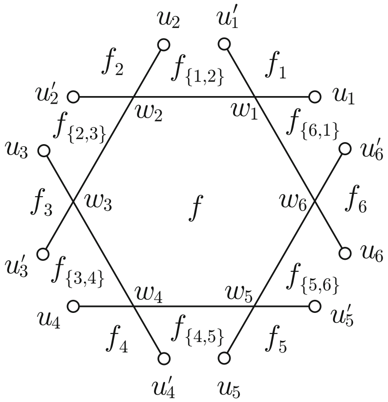

Fig. 1 The case where the component of $(D^{\times}[W])$ is a cycle

$$= -\frac{6}{s} + \frac{\deg(u_i) - 6}{\deg(u_i) - \deg_t(u_i)} + \frac{\deg(u'_i) - 6}{\deg(u'_i) - \deg_t(u'_i)}$$
$$+ \frac{2\deg(f_i) - 6}{\deg(f_i) - \deg_t(f_i)} + \frac{2\deg(f_{\{i,i+1\}}) - 6}{\deg(f_{\{i,i+1\}}) - \deg_t(f_{\{i,i+1\}})}$$
$$+ \frac{2\deg(f_{\{i-1,i\}}) - 6}{\deg(f_{\{i-1,i\}}) - \deg_t(f_{\{i-1,i\}})}.$$

Let

$$\Theta = \sum_{i=1}^{s} \Big(\frac{\deg(u_i) - 6}{\deg(u_i) - \deg_t(u_i)} + \frac{\deg(u'_i) - 6}{\deg(u'_i) - \deg_t(u'_i)} \Big),$$
$$\Phi = \sum_{i=1}^{s} \frac{2\deg(f_i) - 6}{\deg(f_i) - \deg_t(f_i)},$$
$$\Psi = \sum_{i=1}^{s} \Big(\frac{2\deg(f_{\{i,i+1\}}) - 6}{\deg(f_{\{i,i+1\}}) - \deg_t(f_{\{i,i+1\}})} + \frac{2\deg(f_{\{i-1,i\}}) - 6}{\deg(f_{\{i-1,i\}}) - \deg_t(f_{\{i-1,i\}})} \Big).$$

Then

$$\sum_{w \in V(H)} c'(w) = \sum_{i=1}^{s} c'(w_i) = -6 + \Theta + \Phi + \Psi.$$

Since G is a counterexample to Theorem 1.1, we have $\delta(G) \geq 6$ (that is a condition of Theorem 1.1, on Page 2), thus for every true vertex v, we have

$$\frac{\deg(v) - 6}{\deg(v) - \deg_t(v)} \geq 0.$$

By Corollary 2.2, for every big vertex u, we have

$$\frac{\deg(u) - 6}{\deg(u) - \deg_t(u)} \geq 1.$$

Since G is a counterexample to Theorem 1.1, for each i, one of u_i and u'_{i+1} is a big vertex. Thus,

$$\max \Big\{ \frac{\deg(u_i) - 6}{\deg(u_i) - \deg_t(u_i)}, \frac{\deg(u'_{i+1}) - 6}{\deg(u'_{i+1}) - \deg_t(u'_{i+1})} \Big\} \geq 1.$$

Thus $\Theta \geq s$ and then

$$\sum_{w \in V(H)} c'(w) = -6 + \Theta + \Phi + \Psi \geq s - 6 + \Phi + \Psi.$$

Note that, for each i and $f' \in \{f_i, f_{\{i,i+1\}}\}$, $\deg_t(f') \geq 2$. Thus, either $\deg(f') = 3$ or

$$\frac{2\deg(f') - 6}{\deg(f') - \deg_t(f')} \geq 2 - \frac{2}{\deg(f') - 2} \geq 1.$$

It implies that

$$\sum_{w \in V(H)} c'(w) \geq s - 6 + \phi + 2\psi,$$

where ϕ is the number of the faces f_i with $\deg(f_i) \geq 4$ and ψ is the number of the faces $f_{\{i,i+1\}}$ with $\deg(f_{\{i,i+1\}}) \geq 4$.

If $s = 3$ then it is easy to check that every face $f_{\{i,i+1\}}$ has size at least 4 (since two edges of G incident with the same vertex do not cross in D, by [13, Lemma 1.1]), and hence $\psi = 3$ and

$$\sum_{w \in V(H)} c'(w) \geq 3 - 6 + \phi + 3 \geq 0.$$

If $s = 4$ then $\psi \geq 2$, and so

$$\sum_{w \in V(H)} c'(w) \geq 4 - 6 + \phi + 2 \geq 0.$$

If $s \geq 6$ then

$$\sum_{w \in V(H)} c'(w) \geq 6 - 6 + \phi + 2\psi \geq 0.$$

We assume next that $s = 5$. If $\phi + 2\psi \geq 1$ then

$$\sum_{w \in V(H)} c'(w) \geq 5 - 6 + \phi + 2\psi \geq 0.$$

Thus we assume further that $\phi = \psi = 0$, and so $\{u_1, u_2, u_3, u_4, u_5\} = \{u'_1, u'_2, u'_3, u'_4, u'_5\}$. Moreover, it is easy to check that $\{u_1, u_2, u_3, u_4, u_5\}$ contains at least three big vertices. Then

$$\sum_{w \in V(H)} c'(w) = -6 + \Theta = -6 + 2\sum_{i=1}^{5} \frac{\deg(u_i) - 6}{\deg(u_i) - \deg_t(u_i)} \geq -6 + 6 = 0.$$

□

Lemma 2.4 *Let P be a component of $D^{\times}[W]$. Assume that $P := w_1 \cdots w_s$ is a path. Then $c'(w_1) \geq \frac{2}{3}$, $c'(w_s) \geq \frac{2}{3}$ and $c'(w_j) \geq -\frac{1}{3}$ for $2 \leq j \leq s-1$. If further $c'(w_j) < \frac{1}{3}$ and $c'(w_{j+1}) < \frac{1}{3}$ for some j, then either $c'(w_{j+2}) \geq \frac{2}{3}$ or $c'(w_{j-1}) \geq \frac{2}{3}$. In particular,*

$$\sum_{w \in V(P)} c'(w) \geq 0.$$

Proof Let $e_0, e_1, e_2, \ldots, e_s$ be $s+1$ edges of G such that e_{j-1} and e_j cross at w_j where $1 \leq j \leq s$. For $j \in \{1, 2, \ldots, s\}$, denote by y_j and x_{j+1} the two ends of e_j such that w_j is adjacent to x_j and y_j in $D^{\times}$, denote by f_j the face incident with y_j, w_j, x_j, denote by $f_{\{j,j+1\}}$ the face incident with x_j, w_j, w_{j+1}, denote by $f'_{\{j,j+1\}}$ the face incident with w_j, w_{j+1}, w_{j-1} and denote by f'_j the face incident with w_j but other than f_j, $f_{\{j,j+1\}}$ and $f'_{\{j,j+1\}}$, see Fig. 2. (Note that some vertices may be identical.)

Assume that $s = 1$. Then e_1 does not cross edges other than e_0. Recall that for each edge of G, at least one of its ends is big. Then either $\{y_0, y_1, x_1, x_2\}$ contains three big vertices, or $\{y_0, y_1, x_1, x_2\}$ contains exactly two big vertices and w_1 is incident with some face f of $D^{\times}$ which has size at least 4. Note there are at least two true vertices incident to f. Then f sends at least 1 to w_1. Thus we have $c'(w_1) \geq -2 + 3 = 1$.

Assume that $s \geq 2$. For each $1 \leq j \leq s-1$, consider the two faces of $D^{\times}$ incident with $w_j w_{j+1}$, i.e., $f_{\{j,j+1\}}$ and $f'_{\{j,j+1\}}$. Then one of these faces, say $f_{\{j,j+1\}}$, has size at least 4 (since two edges of G incident with the same vertex do not cross in D, by [13, Lemma 1.1]). Moreover, since P is a path, $f_{\{j,j+1\}}$ is incident at least one true vertex. Thus $f_{\{j,j+1\}}$ sends at least $\frac{2\deg(f_{\{j,j+1\}})-6}{\deg(f_{\{j,j+1\}})-1} \geq \frac{2}{3}$ to w_j and w_{j+1}, respectively ($f_{\{j,j+1\}}$ and $f_{\{j-1,j\}}$ may be the same face). Next compute $c'(w_j)$ where $j = 1, 2, \ldots, s$.

Let f_1 and f'_1 be the faces of $D^{\times}$ incident with w_1 than $f_{\{1,2\}}$ and $f'_{\{1,2\}}$. Then either w_1 is adjacent to at least two big vertices or w_1 is adjacent to one big vertex and one of f_1 and f'_1, say f_1, has size at least 4. Noting that $\deg_t(f_1) \geq 2$, we have $c'(w_1) \geq -2 + \frac{2}{3} + 1 + 1 = \frac{2}{3}$. Similarly, we have $c'(w_s) \geq \frac{2}{3}$.

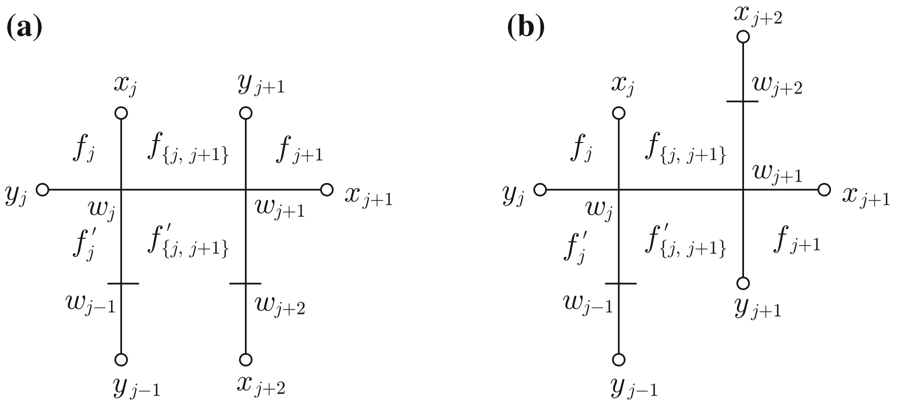

Fig. 2 The case where the component of $(D^{\times}[W])$ is a path

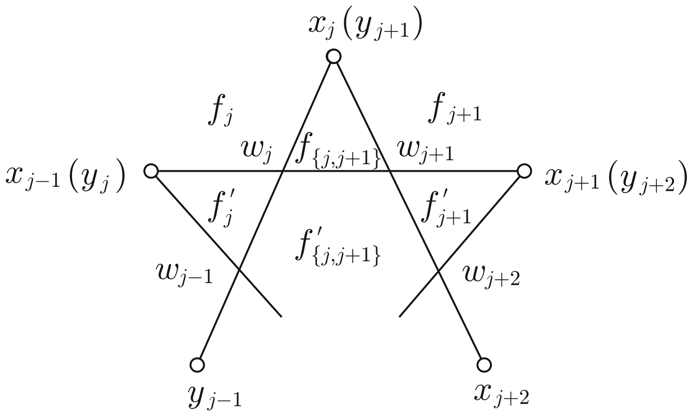

Fig. 3 The case where the sizes of (f'_j), $(f_{\{j,j+1\}})$ and $(f'_{\{j+1\}})$ are three

To complete the proof, we let $s \geq 3$. For each $2 \leq j \leq s-2$, consider w_j. Then either $\deg(f_j) \geq 4$ or one of x_j and y_j is big. Since $\deg_t(f_j) \geq 2$, we know that x_j, y_j and f_j send totally at least 1 to w_j. Thus we have $c'(w_j) \geq -2 + \frac{2}{3} + 1 = -\frac{1}{3}$.

Finally, assume that $c'(w_j) < \frac{1}{3}$ and $c'(w_{j+1}) < \frac{1}{3}$ for some j. Clearly, $2 \leq j \leq s-2$, and there are two cases (a) and (b) as shown in Fig. 2. Consider the case of (b). Both of $f_{\{j,j+1\}}$ and $f'_{\{j,j+1\}}$ send at least $\frac{2}{3}$ to w_j, and x_j, y_j and f_j send totally at least 1 to w_j, thus $c'(w_j) \geq -2 + \frac{2}{3} + \frac{2}{3} + 1 = \frac{1}{3}$, a contradiction. Consider the case of (a). Since P is a path of $D^{\times}$ and $2 \leq j \leq s-2$, $w_{j-1} \neq w_{j+2}$. Thus $\deg(f'_{\{j,j+1\}}) \geq 4$. Again, since P is a path of $D^{\times}$, there is at least one true vertex incident with $f'_{\{j,j+1\}}$. Thus $f'_{\{j,j+1\}}$ sends at least $\frac{2}{3}$ to w_j. Then both of f'_j and $f_{\{j,j+1\}}$ have size 3, otherwise $c'(w_j) \geq \frac{1}{3}$. Similarly, $\deg(f'_{j+1}) = 3$, see Fig. 3.

For the edge $x_{j-1}x_{j+1}$ of D, at least one of x_{j-1} and x_{j+1} is big, and assume that x_{j-1} is big. Then x_j must be small, otherwise $c'(w_j) \geq -2 + \frac{2}{3} + 2 = \frac{2}{3}$, a contradiction. Since x_j is small. y_{j-1} is big. Thus $c'(w_{j-1}) \geq -2 + \frac{2}{3} + 2 = \frac{2}{3}$. Similarly, if x_{j+1} is big, then $c'(w_{j+2}) \geq -2 + \frac{2}{3} + 2 = \frac{2}{3}$.

This completes the proof. □

Now we are ready to get a contradiction. By Lemmas 2.3 and 2.4 , we have

$$\sum_{w \in W} c'(w) = \sum_{H} \sum_{w \in V(H)} c'(w) \geq 0,$$

where H runs over the components of $D^{\times}$. But by (2), $\sum_{w \in W} c'(w) < 0$, a contradiction. This completes the proof of Theorem 1.1.

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