Total monochromatic connection of graphs*

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Abstract

A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a *total monochromatic path* if all the edges and internal vertices on the path have the same color. A total-coloring of a graph is a *total monochromatically-connecting coloring* (*TMC-coloring*, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph G, the *total monochromatic connection number*, denoted by $\text{tmc}(G)$, is defined as the maximum number of colors used in a TMC-coloring of G. These concepts are inspired by the concepts of monochromatic connection number $\text{mc}(G)$, monochromatic vertex connection number $\text{mvc}(G)$ and total rainbow connection number $\text{trc}(G)$ of a connected graph G. Let $l(T)$ denote the number of leaves of a tree T, and let $l(G) = \max\{l(T)|\ T$ is a spanning tree of G $\}$ for a connected graph G. In this paper, we show that there are many graphs G such that $\text{tmc}(G) = m - n + 2 + l(G)$, and moreover, we prove that for almost all graphs G, $\text{tmc}(G) = m-n+2+l(G)$ holds. Furthermore, we compare $\text{tmc}(G)$ with $\text{mvc}(G)$ and $\text{mc}(G)$, respectively, and obtain that there exist graphs G such that $\text{tmc}(G)$ is not less than $\text{mvc}(G)$ and vice versa, and that $\text{tmc}(G) = \text{mc}(G) + l(G)$ holds for almost all graphs. Finally, we prove that $\text{tmc}(G) \leq \text{mc}(G) + \text{mvc}(G)$, and the equality holds if and only if G is a complete graph.

Keywords: total-colored graph, total monochromatic connection, spanning tree with maximum number of leaves

AMS subject classification 2010: 05C15, 05C40, 05C05.

*Supported by NSFC No.11371205 and 11531011, and PCSIRT.

1 Introduction

In this paper, all graphs are simple, finite and undirected. We refer to the book [3] for undefined notation and terminology in graph theory. Throughout this paper, let n and m denote the order (number of vertices) and size (number of edges) of a graph, respectively. Moreover, a vertex of a connected graph is called a *leaf* if its degree is one; otherwise, it is called an *internal vertex*. Let $l(T)$ and $q(T)$ denote the number of leaves and the number of internal vertices of a tree T, respectively, and let $l(G) = \max\{l(T)|\ T$ is a spanning tree of G $\}$ and $q(G) = \min\{q(T)|\ T$ is a spanning tree of G $\}$ for a connected graph G. Note that the sum of $l(G)$ and $q(G)$ is n for any connected graph G of order n. A path in an edge-colored graph is a *monochromatic path* if all the edges on the path have the same color. An edge-coloring of a connected graph is a *monochromatically-connecting coloring* (*MC-coloring*, for short) if any two vertices of the graph are connected by a monochromatic path of the graph. For a connected graph G, the *monochromatic connection number* of G, denoted by $\mathrm{mc}(G)$, is defined as the maximum number of colors used in an MC-coloring of G. An *extremal MC-coloring* is an MC-coloring that uses $\mathrm{mc}(G)$ colors. Note that $\mathrm{mc}(G) = m$ if and only if G is a complete graph. The concept of $\mathrm{mc}(G)$ was first introduced by Caro and Yuster [6] and has been well-studied recently. We refer the reader to [4, 8] for more details.

As a natural counterpart of the concept of monochromatic connection, Cai et al. [5] introduced the concept of monochromatic vertex connection. A path in a vertex-colored graph is a *vertex-monochromatic path* if its internal vertices have the same color. A vertex-coloring of a graph is a *monochromatically-vertex-connecting coloring* (*MVC-coloring*, for short) if any two vertices of the graph are connected by a vertex-monochromatic path of the graph. For a connected graph G, the *monochromatic vertex connection number*, denoted by $\mathrm{mvc}(G)$, is defined as the maximum number of colors used in an MVC-coloring of G. An *extremal MVC-coloring* is an MVC-coloring that uses $\mathrm{mvc}(G)$ colors. Note that $\mathrm{mvc}(G) = n$ if and only if $diam(G) \leq 2$.

Actually, the concepts of monochromatic connection number $\mathrm{mc}(G)$ and monochromatic vertex connection number $\mathrm{mvc}(G)$ are natural opposite concepts of rainbow connection number $\mathrm{rc}(G)$ and rainbow vertex connection number $\mathrm{rvc}(G)$. For details about them we refer the readers to the book [10] and the survey paper [9]. The concept of total rainbow connection number $\mathrm{trc}(G)$ in [12] was motivated by the rainbow connection number $\mathrm{rc}(G)$ and rainbow vertex connection number $\mathrm{rvc}(G)$. Naturally, here we introduce the concept of total monochromatic connection of graphs. A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a *total monochromatic path* if all the edges and internal vertices on the path have the same color. A total-coloring of a graph is a *total monochromatically-connecting coloring* (*TMC-coloring*, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph G, the *total monochromatic connection number*, denoted by $\mathrm{tmc}(G)$, is defined as the maximum number of colors used in a TMC-coloring of G. An *extremal TMC-coloring* is a TMC-coloring that uses $\mathrm{tmc}(G)$ colors. It is easy to check that $\mathrm{tmc}(G) = m + n$ if and only if G is a complete graph.

The rest of this paper is organized as follows: In Section 2, we prove that $\mathrm{tmc}(G) \geq m-n+2+l(G)$ for any connected graph and determine the value of $\mathrm{tmc}(G)$ for some special graphs. In Section 3, we prove that there are many graphs with $\mathrm{tmc}(G) = m-n+2+l(G)$ which are restricted by other graph parameters such as the maximum degree, the diameter and so on. Moreover, we show that for almost all graphs G, $\mathrm{tmc}(G) = m - n + 2 + l(G)$ holds. In Section 4, we compare $\mathrm{tmc}(G)$ with $\mathrm{mvc}(G)$ and $\mathrm{mc}(G)$, respectively, and obtain that there exist graphs G such that $\mathrm{tmc}(G)$ is not less than $\mathrm{mvc}(G)$ and vice versa, and that $\mathrm{tmc}(G) = \mathrm{mc}(G) + l(G)$ for almost all graphs. We also prove that $\mathrm{tmc}(G) \leq \mathrm{mc}(G) + \mathrm{mvc}(G)$, and the equality holds if and only if G is a complete graph.

2 Preliminary results

In this section, we show that $\mathrm{tmc}(G) \geq m - n + 2 + l(G)$ and present some preliminary results on the total monochromatic connection number. Moreover, we determine the value of $\mathrm{tmc}(G)$ when G is a tree, a wheel, and a complete multipartite graph. It is easy to see the following fact.

Proposition 1. *If G is a connected graph and H is a connected spanning subgraph of G, then* $\mathrm{tmc}(G) \geq e(G) - e(H) + \mathrm{tmc}(H)$.

Since for any two vertices of a tree, there exists only one path connecting them, we have the following result.

Proposition 2. *If T is a tree, then* $\mathrm{tmc}(T) = l(T) + 1$.

The consequence below is immediate from Propositions 1 and 2.

Theorem 1. *For a connected graph G,* $\mathrm{tmc}(G) \geq m - n + 2 + l(G)$.

Next we give an important and useful property of an extremal TMC-coloring.

Fact 1. *Let G be a connected graph and f be an extremal TMC-coloring of G that uses a given color c. Then the subgraph H formed by the edges and vertices colored c is a tree whose each internal vertex is colored c.*

Proof. We first claim that H is connected. Otherwise, we will give a fresh color to all the edges and vertices colored c in some component of H while still maintaining a TMC-coloring of G, contradicting the assumption on f. Before proving that H is acyclic, we show that the color of each internal vertex of H is c. Let $u_1, \ldots, u_t$ be the internal vertices of H such that each of them is not colored c. We obtain the subgraph H_0 of H by deleting the vertices $\{u_1, \ldots, u_t\}$. If H_0 is connected, it is possible to choose an edge incident with u_1 in H and assign it with a fresh color while still maintaining a TMC-coloring of G, a contradiction. If not, we can give a fresh color to all the edges and vertices colored c in some component of H_0 while still maintaining a TMC-coloring of G, a contradiction. Now we prove that H does not contain any cycle. Suppose that H has a cycle, say C. Then a fresh color can be assigned to any edge of the cycle C while still maintaining a TMC-coloring of G, which contradicts the assumption on f.

Thus, H is a tree whose each internal vertex is colored c. □

Let G be a connected graph and f be an extremal TMC-coloring of G that uses a given color c. Now we define the *color tree* as the tree formed by the edges and vertices colored c, denoted by T_c. If T_c has at least two edges, the color c is called *nontrivial*. Otherwise, c is *trivial*. We call an extremal TMC-coloring *simple* if for any two nontrivial colors c and d, the corresponding trees T_c and T_d intersect in at most one vertex. If f is simple, then the leaves of T_c must have distinct colors different from color c. Otherwise, we can give a fresh color to such a leaf while still maintaining a TMC-coloring. Moreover, a nontrivial color tree of f with m' edges and q' internal vertices is said to *waste* $m' - 1 + q'$ colors. For the rest of this paper we will use these facts above without further mentioning them.

The lemma below shows that one can always find a simple extremal TMC-coloring for a connected graph.

Lemma 1. *Every connected graph G has a simple extremal TMC-coloring.*

Proof. Given an extremal TMC-coloring f of G with the most number of trivial colors, we prove that this coloring must be simple. Suppose that there exist two nontrivial colors c and d such that T_c and T_d contain k common vertices denoted by $u_1, u_2, \ldots, u_k$, where $k \geq 2$. Now we divide our discussion into two cases.

Case 1. For $1 \leq i \leq k$, u_i is an internal vertex of T_c or T_d.

For $1 \leq i \leq k$, if u_i is an internal vertex of T_c, u_i must be a leaf of T_d and then set $e_i = u_i w_i$ where w_i is the neighbor of u_i in T_d; otherwise, u_i must be a leaf of T_c and then put $e_i = u_i v_i$ where v_i is the neighbor of u_i in T_c. Let H denote the subgraph consisting of the edges and vertices of $T_c \cup T_d$. Clearly, H is connected. We obtain a spanning tree H_0 of H by deleting the edges $\{e_2, e_3, \ldots, e_k\}$. Now we change the total-coloring of H while still maintaining the colors of the leaves in H_0 unchanged. Assign the edges and internal vertices of H_0 with color c and the remaining edges $\{e_2, e_3, \ldots, e_k\}$ with distinct new colors. Obviously, the new total-coloring is also a TMC-coloring and uses $k-2$ more colors than our original one. So, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f.

Case 2. There exists a vertex among $u_1, \ldots, u_k$, say u_1, which is a leaf of both T_c and T_d.

Let v_1 and w_1 be the neighbors of u_1 in T_c and T_d, respectively. There must be another color tree T_e (including a single edge) connecting v_1 and w_1. For $1 \leq i \leq k$, if u_i is a leaf of T_c, then set $e_i = u_iv_i$ where v_i is the neighbor of u_i in T_c; otherwise, u_i must be a leaf of T_d and then put $e_i = u_iw_i$ where w_i is the neighbor of u_i in T_d. Let H_1 denote the subgraph consisting of the edges and vertices of $T_c \cup T_d$. We obtain a spanning subgraph H_2 of H_1 by deleting the edges $\{e_1, e_2, \ldots, e_k\}$. If T_e and H_2 do not have common leaves, let $E_0 = \{e_1, e_2, \ldots, e_k\}$. Otherwise, let $u'_1, \ldots, u'_t$ denote the common leaves of T_e and H_2. Set $e'_i = u'_iv'_i$ where v'_i is the neighbor of u'_i in T_e for $1 \leq i \leq t$. And then let $E_0 = \{e_1, \ldots, e_k, e'_1, \ldots, e'_t\}$. Let H denote the subgraph consisting of the edges and vertices of $T_c \cup T_d \cup T_e$. Clearly, H is connected. We obtain a spanning connected subgraph H_0 of H by deleting the edges of E_0. Now we change the total-coloring of H while still maintaining the colors of the leaves in H_0 unchanged. Assign the edges and internal vertices of H_0 with color c and the remaining edges of H (i.e., the edges of E_0) with distinct new colors. Note that if v is a common leaf of either T_c and T_d or T_e and H_2, it is also a leaf of H_0. Obviously, the new total-coloring is also a TMC-coloring and uses at least $k+t-2$ more colors than our original one. So, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f. □

Now we use the above results to compute the total monochromatic connection numbers of wheel graphs and complete multipartite graphs.

Proposition 3. *Let G be a wheel W_{n-1} of order $n \geq 5$. Then* $\text{tmc}(G) = m-n+2+l(G)$.

Proof. We are given a simple extremal TMC-coloring f of G. Note that $m-n+2+l(G) = m+1$ and $\text{tmc}(G) \geq m+1$ by Theorem 1. Suppose that f consists of k nontrivial color trees, denoted by $T_1, \ldots, T_k$. In fact, we can always find two vertices with degree at least 4 if $k \geq 3$, a contradiction. Likewise, if $k = 2$, G must be W_4 and $\text{tmc}(W_4) = m+1$. Thus, assume that $k = 1$ and T_1 is not spanning (Otherwise, $\text{tmc}(G) = m-n+2+l(G)$). Note that for every vertex $v \notin T_1$, there exist the total monochromatic paths connecting v and the $|T_1|$ vertices of T_1. As f is simple, these paths are internally vertex-disjoint. Hence, $deg(v) \geq |T_1|$. If $|T_1| \geq 4$, the $n-1$ vertices with degree 3 of G must be in T_1 and then T_1 is a path. Thus, $\text{tmc}(G) = m+n-(n-3)-(n-3) = m+6-n \leq m+1$. If

$|T_1| = 3$, then G must be W_3 while $n \geq 5$. Therefore, the proof is completed. $\square$

Proposition 4. *Let* $G = K_{n_1,\ldots,n_r}$ *be a complete multipartite graph with* $n_1 \geq \ldots \geq n_t \geq 2$ *and* $n_{t+1} = \ldots = n_r = 1$. *Then* $\mathrm{tmc}(G) = m + r - t$.

Proof. The case that $r = 2$ is a special case of Theorem 2 whose proof is given in Section 3, so assume that $r \geq 3$. Let f be a simple extremal TMC-coloring of G with maximum trivial colors. Suppose that f consists of k nontrivial color trees, denoted by $T_1, \ldots, T_k$, where $t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Now we divide our discussion into two cases.

Case 1. $t = r$.

In this case, every vertex appears in at least one of the nontrivial color trees. Note that $m - n + 2 + l(G) = m$ and $\mathrm{tmc}(G) \geq m$ by Theorem 1. If $\sum_{i=1}^{k}(t_i - 1) \geq n$, then we have that $\mathrm{tmc}(G) \leq m + n - n - \sum_{i=1}^{k} q_i + k = m - \sum_{i=1}^{k} q_i + k \leq m$. Thus, $\mathrm{tmc}(G) = m$. Suppose that $\sum_{i=1}^{k}(t_i - 1) \leq n - 1$. Now consider the subgraph G' consisting of the union of the T_i and let $C_1, \ldots, C_s$ denote its components.

Now we may assume that there exists a component, say C_1, such that each nontrivial color tree in C_1 is a star. Let S be a star of C_1 with center u and leaves $u_1, \ldots, u_p$, where $u_1, \ldots, u_{p'}$ are in the same vertex class, say V_1. Suppose that $p' \geq 2$. Indeed, if $p' = 1$, we can give a new color to the edge uu_1 while still maintaining a TMC-coloring. We claim that C_1 contains a cycle. If $p' < |V_1|$, there exists a vertex u_{p+1} of V_1 not adjacent to u in S. Then u_1 and u_{p+1} must be in a same nontrivial color tree and the same happens for $u_{p'}$ and u_{p+1}. These nontrivial color trees containing u_1, $u_{p'}$ and u_{p+1} must form a cycle. If $p' = |V_1|$, we have that the vertices of the vertex class containing u must be in a same nontrivial color tree, or we will get a cycle in a similar way. By that analogy, we obtain a cycle formed by some centers of the nontrivial color trees in C_1. Now we change the total-coloring of C_1. We obtain a spanning tree T' of C_1 by connecting u_1 to the vertices in the same class with u and u to the other vertices of C_1. We color the edges and internal vertices of T' with the same color and all other edges and vertices with distinct new colors. Clearly, this new total-coloring is also a TMC-coloring. However, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f.

Thus, suppose that there exists a nontrivial color tree of C_i, say T_{i1}, having two adjacent internal vertices u_i and v_i for $1 \leq i \leq s$. We obtain a spanning tree T by connecting v_1 to each vertex in the same class with u_1 of G and u_1 to the other vertices of G. Now we give a new total-coloring f' of G. Color the edges and internal vertices of T with the same color and all other edges and vertices of G with distinct new colors. Obviously, f' is still a TMC-coloring. If $s \geq 2$, then it either uses more colors or uses the same number of colors but more trivial colors than f, a contradiction. Thus, $s = 1$. Moreover, we can check that f' is a simple extremal TMC-coloring with maximum trivial colors. Therefore, $\text{tmc}(G) = m$.

Case 2. $t < r$.

We obtain a star S by connecting a vertex of $\cup_{i=t+1}^{r} V_i$ to each vertex of $\cup_{i=1}^{t} V_i$. Color the edges and the center vertex of S with the same color and all other edges and vertices of G with distinct new colors. Clearly, this new total-coloring is still a TMC-coloring, denoted by f'. Thus, $\text{tmc}(G) \geq m + r - t$. If $\sum_{i=1}^{k}(t_i - 1) \geq n - r + t$, then we have that $\text{tmc}(G) \leq m + n - (n - r + t) - \sum_{i=1}^{k} q_i + k = m + r - t - \sum_{i=1}^{k} q_i + k \leq m + r - t$. Hence, $\text{tmc}(G) = m + r - t$. Suppose that $\sum_{i=1}^{k}(t_i - 1) \leq n - r + t - 1$. Next consider the subgraph G' consisting of the union of the T_i's and suppose that it has s components, say $C_1, \ldots, C_s$. Note that $|V(G')| \geq n - r + t$ since any two vertices of the same class must be covered in a nontrivial color tree. The case that $|V(G')| = n - r + t$ can be verified by a similar discussion to Case 1. Thus, suppose that $|V(G')| > n - r + t$. It is obvious that $s \geq 2$. Moreover, there must exist a vertex x of $\cup_{i=t+1}^{r} V_i$, which is contained in a component of G', say C_1. For $2 \leq j \leq s$, there does not exist a vertex of $\cup_{i=t+1}^{r} V_i$ in C_j. Otherwise, let x be the center of S and then f' either uses more colors or uses the same number of colors but more trivial colors than f, a contradiction. By a similar discussion to Case 1, we can obtain that there exists a nontrivial color tree of C_j having two adjacent internal vertices for $2 \leq j \leq s$. We obtain a star S_1 by joining the vertices of $\cup_{i=2}^{s} C_i$ to one internal vertex of C_1. We give a new total-coloring of G while still maintaining the total-coloring of C_1 unchanged. Assign the edges and the center vertex of S_1 with one color and the other edges and vertices of $G \backslash C_1$ with distinct new colors. This new total-coloring is still a TMC-coloring and it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f. Therefore, we have finished the proof. □

3 Graphs with $\text{tmc}(G) = m - n + 2 + l(G)$

In this section, we prove that there are many graphs G for which $\text{tmc}(G) = m - n + 2 + l(G)$, and also show that the equality holds for almost all graphs.

Lemma 2. *[6] Let G be a connected graph of order $n > 3$. If G satisfies any of the following properties, then* $\text{mc}(G) = m - n + 2$.

(a) *The complement $\overline{G}$ of G is 4-connected.*

(b) *G is K_3-free.*

(c) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. *In particular, this holds if $\Delta(G) \leq (n + 1)/2$, and this also* *holds if* $\Delta(G) \leq n - 2m/n$.

(*d*) *diam*$(G) \geq 3$.

(*c*) *G has a cut vertex.*

We can obtain that $\text{tmc}(G) \leq \text{mc}(G) + l(G)$ for a noncomplete graph, whose proof is contained in the proof of Theorem 6 in Section 4. Together with Theorem 1 and Lemma 2, we have the following results.

Theorem 2. *Let G be a connected graph of order* $n > 3$. *If G satisfies any of the following properties, then* $\text{tmc}(G) = m - n + 2 + l(G)$.

(*a*) *The complement* $\overline{G}$ *of G is* 4*-connected.*

(*b*) *G is* K_3*-free.*

(*c*) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$.

(*d*) *diam*$(G) \geq 3$.

(*c*) *G has a cut vertex.*

The upper bound of $\Delta(G)$ in Theorem 2(c) is the best possible. For example, let $G = K_{n-2,1,1}$. Then $\text{tmc}(G) = m - n + 3 + l(G)$ and $\Delta(G) = n - 1 = n - \frac{2m-3(n-1)}{n-3}$.

From Theorem 2(a), we can get a stronger result. For a property P of graphs and a positive integer n, define $Prob(P, n)$ to be the ratio of the number of graphs with n labeled vertices having P over the total number of graphs with these vertices. If $Prob(P, n)$ approaches 1 as n tends to infinity, then we say that *almost all* graphs have the property P. See [1] for example.

Theorem 3. *For almost all graphs G, we have that* $\text{tmc}(G) = m - n + 2 + l(G)$.

In order to prove Theorem 3, we need the following lemma.

Lemma 3. *[1] For every nonnegative integer k, almost all graphs are k-connected.*

Proof of Theorem 3: For any given nonnegative integer n, let $\mathscr{G}_n$ denote the set of all graphs of order n, and let $\mathscr{G}_n^4$ denote the set of all 4-connected graphs of order n. Moreover, let $\mathscr{B}_n$ denote the set of all graphs G of order n such that the complement $\overline{G}$ of G is 4-connected. Note that for any two graphs G and H, $G \cong H$ if and only if $\overline{G} \cong \overline{H}$. Then, it is easy to check that the map: $G \to \overline{G}$ is a bijection from $\mathscr{B}_n$ to $\mathscr{G}_n^4$. Therefore, we have

$$\frac{|\mathscr{B}_n|}{|\mathscr{G}_n|} = \frac{|\mathscr{G}_n^4|}{|\mathscr{G}_n|}.$$

By Lemma 3, it follows that almost all graphs are 4-connected. Then, we get that almost all graphs have 4-connected complements. Furthermore, since almost all graphs are connected, we have that $\text{tmc}(G) = m - n + 2 + l(G)$ by Theorem 2(a). □

Remark 1. For the monochromatic connection number $\mathrm{mc}(G)$, from Lemma 2(a) and Lemma 3, one can deduce, in a similar way, that for almost all graphs G, $\mathrm{mc}(G) = m-n+2$ holds.

Remark 2. Although the parameter $l(G)$ seems nice in the expression of the lower bound of $\mathrm{tmc}(G)$, from [7, p.206], it is NP-hard to find a spanning tree with $l(G)$ leaves in a connected graph G.

4 Comparing $\mathrm{tmc}(G)$ with $\mathrm{mvc}(G)$ and $\mathrm{mc}(G)$

Let G be a nontrivial connected graph. Firstly, we compare $\mathrm{tmc}(G)$ with $\mathrm{mvc}(G)$. The question we may ask is, can we bound one of $\mathrm{tmc}(G)$ and $\mathrm{mvc}(G)$ in terms of the other? The following two theorems give sufficient conditions for $\mathrm{tmc}(G) > \mathrm{mvc}(G)$.

Theorem 4. *Let G be a connected graph with diameter d. If $m \geq 2n-d-2$, then* $\mathrm{tmc}(G) > \mathrm{mvc}(G)$.

Proof. The case that $d = 1$ is trivial, so assume that $d \geq 2$. We can check that if $l(G) = 2$, then $\mathrm{tmc}(G) > \mathrm{mvc}(G)$. Thus, suppose that $l(G) \geq 3$. By Theorem 1, it follows that $\mathrm{tmc}(G) \geq m-n+2+l(G) \geq 2n-d-2-n+2+3 = n-d+3$. Moreover, we have that $\mathrm{mvc}(G) \leq n-d+2$ by [5, Proposition 2.3]. Therefore, $\mathrm{tmc}(G) > \mathrm{mvc}(G)$. □

Theorem 5. *Let G be a connected graph of diameter 2 with maximum degree Δ. If $\Delta \geq \frac{n+1}{2}$, then* $\mathrm{tmc}(G) > \mathrm{mvc}(G)$.

Before proving Theorem 5, we need the lemma below.

Lemma 4. *[2] Let G be a connected graph of diameter 2 with maximum degree Δ. Then*

$$m \geq \begin{cases} n+\Delta-2, & if\ \Delta = n-2 \text{ or } n-3 \\ 2n-5, & if\ \Delta = n-4 \\ 2n-4, & if\ \frac{2n-2}{3} \leq \Delta \leq n-5 \\ 3n-\Delta-6, & if\ \frac{3n-3}{5} \leq \Delta < \frac{2n-2}{3} \\ 5n-4\Delta-10, & if\ \frac{5n-3}{9} \leq \Delta < \frac{3n-3}{5} \\ 4n-2\Delta-11, & if\ \frac{n+1}{2} \leq \Delta < \frac{5n-3}{9} \end{cases} \tag{1}$$

Proof of Theorem 5: The case that $n \leq 7$ can be easily verified. Suppose that $n \geq 8$. Since the diameter of G is 2, we have that $\mathrm{mvc}(G) = n$. By Theorem 1 and Lemma 4, $\mathrm{tmc}(G) \geq m-n+2+l(G) > n$. Thus, $\mathrm{tmc}(G) > \mathrm{mvc}(G)$. □

Actually, we have that $\mathrm{tmc}(C_5) = 4 < \mathrm{mvc}(C_5) = 5$, where $m < 2n-d-2$ and $\Delta < \frac{n+1}{2}$. This implies that the conditions of Theorems 4 and 5 cannot be improved. Moreover, if G is a star, then $\mathrm{tmc}(G) = \mathrm{mvc}(G) = n$. Therefore, there exist graphs G such that $\mathrm{tmc}(G)$ is not less than $\mathrm{mvc}(G)$ and vice versa. However, we cannot show whether there exist other graphs with $\mathrm{tmc}(G) \leq \mathrm{mvc}(G)$. Thus, we propose the following problem.

Problem 1. *Dose there exist a graph of order $n \geq 6$ except a star such that $\mathrm{tmc}(G) \leq \mathrm{mvc}(G)$?*

Next we compare $\mathrm{tmc}(G)$ with $\mathrm{mc}(G)$. If G satisfies one of the conditions in Theorem 2, then we have $\mathrm{mc}(G) = m-n+2$ and so $\mathrm{tmc}(G) = \mathrm{mc}(G)+l(G)$. For a complete graph G, $\mathrm{tmc}(G) > \mathrm{mc}(G) + l(G)$. From [6, Corollary 13], if G is a wheel W_{n-1} of order $n \geq 5$, we have that $\mathrm{mc}(G) = m - n + 3$ and then $\mathrm{tmc}(G) < \mathrm{mc}(G) + l(G)$. However, by Theorem 3 and Remark 1, it follows that almost all graphs have that $\mathrm{tmc}(G) = \mathrm{mc}(G) + l(G)$ which implies that almost all graphs have that $\mathrm{tmc}(G) > \mathrm{mc}(G)$. Thus, we propose the following conjecture.

Conjecture 1. For a connected graph G, it always holds that $\mathrm{tmc}(G) > \mathrm{mc}(G)$.

Finally, we compare $\mathrm{tmc}(G)$ with $\mathrm{mc}(G) + \mathrm{mvc}(G)$.

Theorem 6. *Let G be a connected graph. Then $\mathrm{tmc}(G) \leq \mathrm{mc}(G) + \mathrm{mvc}(G)$, and the equality holds if and only if G is a complete graph.*

In order to prove Theorem 6, we need the following lemma.

Lemma 5. *For a noncomplete connected graph G, let f be a simple extremal TMC-coloring of G and $T_1, \ldots, T_k$ denote all the nontrivial color trees of f, where $t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Then, $\sum_{i=1}^{k} q_i \geq q(G)$.*

Proof. For any $v \in G$, if $v \notin \cup_{i=1}^{k} T_i$, v must be adjacent to an internal vertex w_0 of a nontrivial color tree and then set $E_v = \{vw | w \in N(v) \backslash \{w_0\}\}$. If v is an internal vertex of a nontrivial color tree containing v, set $E_v = \emptyset$. Otherwise, v is a leaf of any nontrivial color tree containing v. Let $T_1, \ldots, T_s$ denote the nontrivial color trees containing v and $v_1, \ldots, v_s$ be the neighbors of v in $T_1, \ldots, T_s$, respectively. Let $E_v = \{vv_2, \ldots, vv_s\}$. We obtain a spanning subgraph G' by deleting the edges of $\bigcup_{v \in G} E_v$. Note that every vertex of $\{v : E_v = \emptyset\}$ is connected to each other. For any two vertices u_1 and u_2 of $\{v : E_v = \emptyset\}$, there exists a total monochromatic path P of G connecting them. For each vertex u of P, we have $E_u = \emptyset$. Thus, G' also contains P from u_1 to u_2. Moreover, every vertex of $\{v : E_v \neq \emptyset\}$ is connected to a vertex of $\{v : E_v = \emptyset\}$. Hence, G' is connected and each vertex of $\{v : E_v \neq \emptyset\}$ cannot be an internal vertex of G'. Then $\sum_{i=1}^{k} q_i \geq q(G') \geq q(G)$. □

Now, we are ready to prove Theorem 6.

Proof of Theorem 6: If G is a complete graph, we have that $\mathrm{tmc}(G) = \mathrm{mc}(G) + \mathrm{mvc}(G)$. Thus, suppose that G is not complete. We are given a simple extremal TMC-coloring f of G. Suppose that f consists of k nontrivial color trees denoted by $T_1, \ldots, T_k$, where $t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Then $\mathrm{tmc}(G) = m + n - \sum_{i=1}^{k}(t_i - 2) - \sum_{i=1}^{k} q_i$. Now we take a copy G' of G. Then G' contains the trees $T'_1, \ldots, T'_k$ corresponding to $T_1, \ldots, T_k$, respectively. Define an edge-coloring f_e of G' as follows: color the edges of T_i with color i for $1 \leq i \leq k$ and the other edges of G' with distinct new colors. Then f_e is an MC-coloring of G' with $m - \sum_{i=1}^{k}(t_i - 2)$ colors. Thus, $\mathrm{mc}(G) = \mathrm{mc}(G') \geq m - \sum_{i=1}^{k}(t_i - 2) = \mathrm{tmc}(G) - n + \sum_{i=1}^{k} q_i$. By Lemma 5, we have that $\sum_{i=1}^{k} q_i \geq q(G)$. Then $\mathrm{tmc}(G) \leq \mathrm{mc}(G) + n - q(G) = \mathrm{mc}(G) + l(G)$. Moreover, it is easy to obtain that $\mathrm{mvc}(G) \geq l(G) + 1$. Hence, $\mathrm{tmc}(G) < \mathrm{mc}(G) + \mathrm{mvc}(G)$. Therefore, the proof is completed. □

Remark 3. For the total rainbow connection number $\mathrm{trc}(G)$, we cannot bound one of $\mathrm{trc}(G)$ and $\mathrm{rc}(G) + \mathrm{rvc}(G)$ in terms of the other. For a connected graph G, $\mathrm{trc}(G) = \mathrm{rc}(G) + \mathrm{rvc}(G)$ if G is a complete graph or a star. Moreover, if G is a complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 6^m$, then $\mathrm{trc}(G) = 7 > \mathrm{rc}(G) + \mathrm{rvc}(G) = 4 + 1$ [9, 11, 12]. In [12], for every $s \geq 1481$, there exists a graph G with $\mathrm{trc}(G) = \mathrm{rvc}(G) = s$ which implies that $\mathrm{trc}(G) < \mathrm{rc}(G) + \mathrm{rvc}(G)$. This is one thing that the total monochromatic connection differs from the total rainbow connection.

Acknowledgement: The authors are very grateful to the reviewers and editor for their useful suggestions and comments, which help us improve the presentation of the paper.

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