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Improved upper bound for the degenerate and star chromatic numbers of graphs

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Abstract Let $G = G(V, E)$ be a graph. A proper coloring of G is a function $f : V \to N$ such that $f(x) \neq f(y)$ for every edge $xy \in E$. A proper coloring of a graph G such that for every $k \geq 1$, the union of any k color classes induces a $(k-1)$-degenerate subgraph is called a degenerate coloring; a proper coloring of a graph with no two-colored P_4 is called a star coloring. If a coloring is both degenerate and star, then we call it a degenerate star coloring of graph. The corresponding chromatic number is denoted as $\chi_{sd}(G)$. In this paper, we employ entropy compression method to obtain a new upper bound $\chi_{sd}(G) \leq \lceil \frac{19}{6}\Delta^{\frac{3}{2}} + 5\Delta \rceil$ for general graph G.

Keywords Degenerate coloring · Star coloring · Chromatic number · Entropy compression method · Upper bound

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A k-coloring of graph G is an function $f : V \to N$ such that $|f(V)| = k$. The coloring f of G is called proper if $f(x) \neq f(y)$ for every edge $xy \in E$. A degenerate coloring of graph G is a proper coloring such that for every $k \geq 1$, the union of any k color classes induces a $(k-1)$-degenerate subgraph. A star coloring of a graph G is a proper coloring such

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that for the union of any two color classes induces a star forest. If a coloring of G is both degenerate and star, then it is called a degenerate and star coloring of G, the corresponding chromatic number is denoted by $\chi_{sd}(G)$.

Obviously, the notion of a degenerate coloring is a strengthening of the notion of an acyclic coloring. A graph G is k-degenerate if every subgraph of G has a vertex of degree less than or equal to k. A coloring of a graph such that for every $k \geq 1$, the union of any k color classes induces a $(k-1)$-degenerate subgraph is a degenerate coloring.

As for the degenerate coloring of planar graphs, there is a Conjecture proposed by Borodin.

Conjecture (Borodin 1979) *Every planar graph can be colored with five colors, so that the union of any k-color classes induces a* $(k-1)$*-degenerate graph for* $k = 1, \ldots, 4$.

Rautenbach (2008) proved that the existence of degenerate colorings of planar graphs using eighteen colors. Their result is as follows.

Theorem 1 (Rautenbach 2008) *For any planar graph, the degenerate colorings of planar graphs using eighteen colors such that the union of any k color classes induces a* $(k-1)$*-degenerate graph for* $k = 1, \ldots, 5$.

This result was improved in Mohar and Špacapan (2009).

Theorem 2 (Kierstead et al. 2009) *For any planar graph, the degenerate colorings of planar graphs using nine colors such that the union of any k color classes induce a* $(k-1)$*-degenerate graph.*

For nonplanar graphs, Mohar and Špacapan (2012) used *Lovász Local Lemma* to give an upper bound for list version result of degenerate and star chromatic number.

Theorem 3 (Mohar and Špacapan 2012) *For any graph with maximum degree* Δ*, there is a degenerate star list coloring of G whenever the list of each vertex contains at least* $\lceil 1000\Delta^{\frac{3}{2}} \rceil$ *admissible colors.*

As for the star chromatic number of a graph G, (denoted by $\chi_s(G)$), Fertin et al. (2004) proved that for every graph G with maximum degree Δ, $\chi_s(G) \leq 20\Delta^{\frac{3}{2}}$, and that this bound is best possible up to a polylogarithmic factor: for some absolute constant C, there are graphs with maximum degree Δ requiring $\frac{C\Delta^{\frac{3}{2}}}{(log\Delta)^{\frac{1}{2}}}$ colors in any star coloring. Ndreca et al. (2012) showed that for every graph G with maximum degree Δ, $\chi_s(G) \leq 4.34\Delta^{\frac{3}{2}} + 1.5\Delta$. So according to the above results, the upper bound of degenerate star chromatic number of Theorem 3 is best possible up to a polylogarithmic factor.

Recently Moser and Tardos (2010) designed an algorithmic version of *Lovász Local Lemma* by means of the so-called *Entropy Compression Method*. Using this method, Esperet and Parreau gave an improved upper bound for acyclic edge-coloring and star coloring of graphs in Esperet and Parreau (2013). Goncalves et al. (2014) provide a more general method and give new tool to improve the analysis.

In this paper, we employ Entropy Compression Method to improve the degenerate and star chromatic number to $\chi_{sd}(G) \leq \lceil \frac{19}{6}\Delta^{\frac{3}{2}} + 5\Delta \rceil$. Actually, we obtain the following result.

Theorem 4 *For any graph G with maximum degree* Δ*, there is a degenerate star coloring of G with* $\lceil \frac{19}{6}\Delta^{\frac{3}{2}} + 5\Delta \rceil$ *colors such that for every vertex v of degree at most* $\Delta^{\frac{1}{2}}$*, all neighbors of v are colored differently.*

2 Preliminary

In this section, we will make some preparations for the proof of Theorem 4. Let $G = (V, E)$ be a graph with vertex set V and edge set E. Under the coloring f of G, first we define a family F of subgraph of G under coloring f. In Fig. 1 we show a set of subgraphs F of G, for every subgraph $R \in F$ of G, there is a coloring f_R (which is shown in Fig. 1) is given under coloring f.

For $X, Y \subseteq V$, we use $E(X, Y)$ to denote the set of edges whose one endvertex in X and the other in Y. We present the following Observation and Lemma given by Mohar and Špacapan in (2012). We include the proof of the Observation 2.1 and Lemma 2.2 for completeness.

Observation 2.1 (Mohar and Špacapan 2012) *Let G be a graph with minimum degree* $k \geq 2$ *and let f be a proper k-coloring of G. If S is a non-empty subset of a color class* C_j *of f, then there exists a color class* C_i *of f, such that* $|E(S, C_j)| \geq \frac{k}{k-1}|S| > |S|$.

Proof Each vertex in S has degree at least k. Therefore,

$$\sum_{j \neq i} |E(S, C_j)| \geq k|S|,$$

which implies the claimed inequalities. □

Lemma 2.2 (Mohar and Špacapan 2012) *If no event of subgraphs of F in Fig. 1 of G under coloring f occurs, then the coloring f of G is a degenerate star coloring such that for every vertex of degree* $\leq \Delta^{\frac{1}{2}}$ *all its neighbors are colored by pairwise different colors.*

Proof In this proof, if ith of the subgraph under coloring of f in Fig. 1 occurs, we will call it event type i occurs. Since no event of type (2), (8) or (10) occurs, the coloring f is a star coloring such that any vertex of degree at most $\Delta^{\frac{1}{2}}$ has its neighbors colored by different colors. It remains to prove that the coloring is degenerate. Suppose on the contrary, that there is a subgraph Q of G with minimum degree k colored by k colors. Since f is a proper coloring, we have $k \geq 2$. Then there exists a vertex $x \in V(Q)$ of (say) color 1 adjacent to two vertices y, z of color 2 (see Observation 2.1). Furthermore, there is a color class P of f such that $|E(\{y, z\}, P)| \geq 3$. Since events of type 2 do not occur, the color of P is not 1 or 2. Similarly we see that $N(y) \cap N(z) \cap P = \emptyset$.

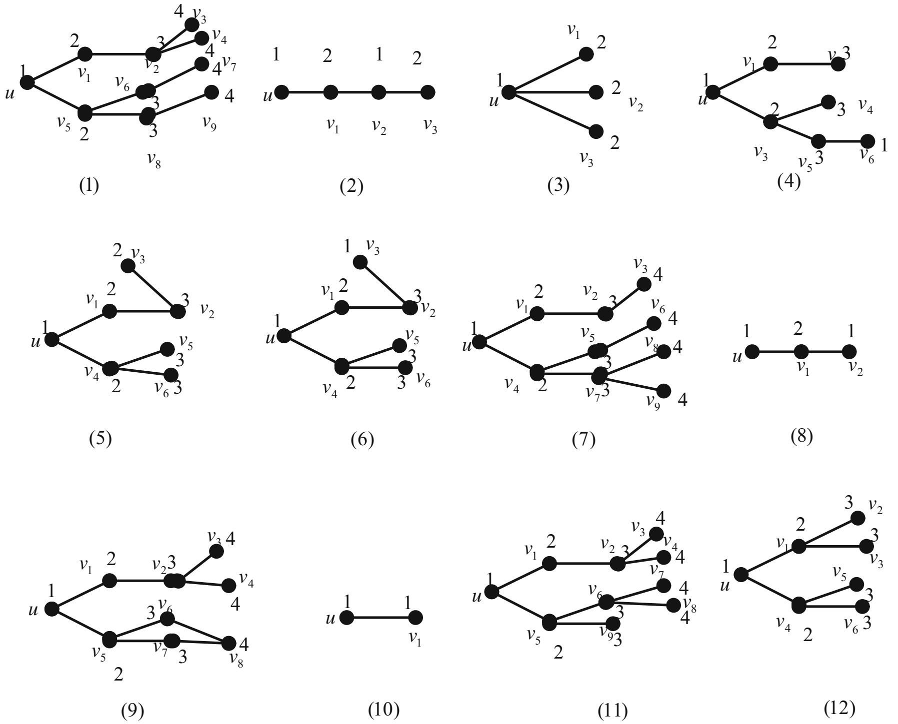

Fig. 1 The set of subgraphs F= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} of G under coloring f

As events of type (3) are excluded, there are three vertices $u, v, w \in P$, such that u, v are adjacent to y and w is adjacent to z. Let $Y = \{u, v, w\}$. Then there is a color class $P^{'}$ such that $|E(Y, P^{'})| \geq 4$. Since events of type (2), (4), (5), and (6) do not occur, $P^{'}$ is distinct from the color classes 1 and 2. If $|N(Y) \cap P^{'}| \geq 4$ then (since (2) and (3) do not occur) an event of type (7), (1), (11), (12) happens, a contradiction. If $|N(Y) \cap P^{'}| \leq 3$, then a similar argument shows that either type (9) or (5) event occurs. This contradiction proves Lemma 2.2. □

3 The proof of Theorem 4

We prove Theorem 4 by contradiction. Suppose that there exists a graph G satisfying the conditions of Theorem 4 such that $\chi_{sd}(G) \geq \lceil \frac{19}{6}\Delta^{\frac{3}{2}} + 5\Delta \rceil + 1$. Let $\kappa = \lceil \frac{19}{6}\Delta^{\frac{3}{2}} + 5\Delta \rceil$. We employ an algorithm to "manage" to color G which guarantee the coloring is a degenerate star coloring with κ colors. We define a total order $\prec$ on the vertices of G. In the following, we first define an algorithm, then we give an analysis of the algorithm, finally we obtain a contradiction.

We first give an algorithm to partially color G that guarantee the coloring is degenerate and star.

3.1 The algorithm

Let $M = \{1, 2, \ldots, \kappa\}^t$ be the vector of length t, for some arbitrarily large $t \gg n = |V(G)|$. Algorithm DegeneratestarColoring takes the vector M as input and returns a partial degenerate star coloring $\varphi : V(G) \rightarrow \{\bullet, 1, 2, \ldots, \kappa\}$ of G, ($\bullet$ means that the vertex is uncolored) and a file, called record R. The degenerate and star coloring φ is necessarily partial since we actually try to color G with a number of colors less than its degenerate star chromatic number. Through discussion in Sect. 2, we know that if no event of Fig. 1 occurs in the coloring process, the coloring remains a partial degenerate and star coloring. For this reason, we call the event 1–12 bad events.

Algorithm DegeneratestarColoring runs as follows. Let φ_i be the partial coloring of G after i steps. At step i, we examine φ_{i-1} and color the smallest uncolored vertex v by $M[i]$, after that we verify whether one of the 12 bad events happens.

If one of such events happens, then we uncolor some vertices in order that none of the above mentioned 12 events happens. Through this process, we can modify the coloring such that none of the 12 events remains. According to Lemma 2.2, we can see that φ_i is a partial degenerate star coloring of G.

In the following, we first prove that the function defined in algorithm DegeneratestarColoring is injective. Then we will obtain a contradiction since we can show that the number of possible outputs is strictly smaller than the number of possible inputs when t is much larger than n. Actually, in the following we can see that the number of possible inputs is exactly κ^t while the number of possible outputs is $o(\kappa^t)$. Therefore, we can obtain the desired contradiction and complete the proof of Theorem 4.

3.2 Analysis of algorithm DegeneratestarColoring

Recall that φ_i denotes the partial degenerate star coloring obtained after i steps. Let us denote by $\overline{\varphi}_i \subset V(G)$ the set of vertices that are colored in φ_i. Let also v_i, R_i and M_i respectively denote the current vertex v of the i^{th} step, the record R after i steps, and the input vector M restricted to its i first elements. Observe that as φ_i is a partial degenerate star κ-coloring of G, and as G is not degenerate star κ-colorable, we have that $\overline{\varphi}_i \subsetneq V(G)$, and thus v_{i+1} is well defined. This also implies that R has t "`Color`" lines. Finally observe that R_i corresponds to the lines of R before the $(i+1)^{\text{th}}$ "`Color`" line.

Algorithm DegeneratestarColoring_G: Part 1

Input : M (vector of length t).
Output : (φ, R).

for all $v \in V(G)$ **do**
    $\varphi(v) \leftarrow \bullet$
end for
$R \leftarrow newfile()$
for all $i \leftarrow 1$ **to** t **do**
    Let v be the smallest (w.r.t. $\prec$) uncolored vertex of G
    $\varphi(v) \leftarrow M[i]$
    Write "`Color` \n" in R
    if Event 1 happens **then** `Event 1 issue`
    $\varphi(v_4) \leftarrow \bullet$
    $\varphi(v_5) \leftarrow \bullet$
    $\varphi(v_6) \leftarrow \bullet$
    $\varphi(v_7) \leftarrow \bullet$
    $\varphi(v_8) \leftarrow \bullet$
    $\varphi(v = v_9) \leftarrow \bullet$
    Write "`Uncolor, event 1` $(v_4, v_5, v_6, v_7, v_8, v_9)$ \n" in R
    else if Event 2 happens **then** `Event 2 issue`
    $\varphi(v_2) \leftarrow \bullet$
    $\varphi(v = v_3) \leftarrow \bullet$
    Write "`Uncolor, event 2` (v_2, v_3) \n" in R
    else if Event 3 happens **then** `Event 3 issue`
    $\varphi(v_2) \leftarrow \bullet$
    $\varphi(v = v_3) \leftarrow \bullet$
    Write "`Uncolor,event 3` (v_2, v_3) \n" in R
    else if Event 4 happens **then** `Event 4 path issue`
    $\varphi(v_3) \leftarrow \bullet$
    $\varphi(v_4) \leftarrow \bullet$
    $\varphi(v_5) \leftarrow \bullet$
    $\varphi(v = v_6) \leftarrow \bullet$
    Write "`Uncolor,event 4` (v_3, v_4, v_5, v_6) \n" in R
    else if Event 5 happens **then** `Event 5 issue`
    $\varphi(v_3) \leftarrow \bullet$
    $\varphi(v_4) \leftarrow \bullet$
    $\varphi(v_5) \leftarrow \bullet$
    $\varphi(v = v_6) \leftarrow \bullet$
    Write "`Uncolor,event 5` (v_3, v_4, v_5, v_6) \n" in R
    else if Event 6 happens **then** `Event 6 issue`
    $\varphi(v_3) \leftarrow \bullet$
    $\varphi(v_4) \leftarrow \bullet$
    $\varphi(v_5) \leftarrow \bullet$
    $\varphi(v = v_6) \leftarrow \bullet$
    Write "`Uncolor,event 6` (v_3, v_4, v_5, v_6) \n" in R

Algorithm DegeneratestarColoring_G: Part 2

13: **else if** Event 7 happens **then** `Event 7 issue`
$\varphi(v_4) \leftarrow \bullet$
$\varphi(v_5) \leftarrow \bullet$
$\varphi(v_6) \leftarrow \bullet$
$\varphi(v_7) \leftarrow \bullet$
$\varphi(v_8) \leftarrow \bullet$
$\varphi(v = v_9) \leftarrow \bullet$
Write "`Uncolor,event 7` $(v_4, v_5, v_6, v_7, v_8, v_9)$ \n" in R
14: **else if** Event 8 happens **then** `Event 8 issue`
$\varphi(v = v_2) \leftarrow \bullet$
Write "`Uncolor,event 8` (v_2) \n" in R
15: **else if** Event 9 happens **then** `Event 9 issue`
$\varphi(v_4) \leftarrow \bullet$
$\varphi(v_5) \leftarrow \bullet$
$\varphi(v_6) \leftarrow \bullet$
$\varphi(v_7) \leftarrow \bullet$
$\varphi(v = v_8) \leftarrow \bullet$
Write "`Uncolor,event 9` $(v_4, v_5, v_6, v_7, v_8)$ \n" in R
16: **else if** Event 10 happens **then** `Event 10 issue`
$\varphi(v = v_1) \leftarrow \bullet$
Write "`Uncolor,event 10` (v_1) \n" in R
17: **else if** Event 11 happens **then** `Event 11 issue`
$\varphi(v_4) \leftarrow \bullet$
$\varphi(v_5) \leftarrow \bullet$
$\varphi(v_6) \leftarrow \bullet$
$\varphi(v_7) \leftarrow \bullet$
$\varphi(v_8) \leftarrow \bullet$
$\varphi(v = v_9) \leftarrow \bullet$
Write "`Uncolor,event 11` $(v_4, v_5, v_6, v_7, v_8, v_9)$ \n" in R
18: **else if** Event 12 happens **then** `Event 12 issue`
$\varphi(v_3) \leftarrow \bullet$
$\varphi(v_4) \leftarrow \bullet$
$\varphi(v_5) \leftarrow \bullet$
$\varphi(v = v_6) \leftarrow \bullet$
Write "`Uncolor,event 12` (v_3, v_4, v_5, v_6) \n" in R
19: **end if**
20: **end for**
return (φ, R)

Let us first show that the function defined in algorithm DegeneratestarColoring is injective.

Lemma 3.1 *One can recover M_i from (φ_i, R_i).*

Proof First note that at every step of algorithm DegeneratestarColoring, a line "Color" may possibly followed by a line "Uncolor" which is appended to R. Then we call the step which only appends a line "Color" a $colorstep$, while the step appends a line "Color" followed by a line "Uncolor" step an $Uncolorstep$. Hence, by looking at the last line of R, we can know whether the last step is a $colorstep$ or an $uncolorstep$.

We first show that R_i uniquely determines the set of colored vertices at step i by induction on i. It is easy to see that R_1 contains only one "Color" line, then v_1 is the unique colored vertex at step 1. Assume that $i \geq 2$. By induction hypothesis,

R_{i-1} uniquely determines the set of colored vertices at step $i-1$. At step i, color the smallest uncolored vertex of G. If the last line of R_i is a $Color$ line, then we can know $\overline{\varphi}_i$ easily; if one of Event 1–12 occurs, then the last line of R_i is an $Uncolor$ line which indicates the vertices that uncolored. Hence, R_i uniquely determines the set of colored vertices at step i.

Now we prove that M_i can be recovered from the pair (φ_i, R_i) by induction on i. First we know that at step 1, M_1 can be recovered from the pair (φ_1, R_1) since v_1 is the unique colored vertex. In this case $M[1] = \varphi_1(v_1)$. Assume that $i \geq 2$. According to the discussion above, the record R_{i-1} indicates the set of colored vertices at step $i-1$, so we know that the smallest uncolored vertex v at the beginning of step i.

If step i was a color step, then one can obtain φ_{i-1} from φ_i in such a way that $\varphi_{i-1}(u) = \varphi_i(u)$ for $u \neq v$ and $\varphi_{i-1}(v) = \bullet$. By induction hypothesis, one can recover M_{i-1} from (φ_{i-1}, R_{i-1}) and $M[i] = \varphi_i(v)$. Therefore, we can recover M_i from the pair (φ_i, R_i).

If step i was an uncolor step, then by the last line of R_i and the above discussion, we can determine the set of colored vertices at step i and deduce φ_{i-1}. Then by induction hypothesis, M_{i-1} can be recovered from (φ_{i-1}, R_{i-1}). Hence, we can obtain M_i by considering the following cases:

- If the last line is "Uncolor, event 1", then $M[i] = \varphi_i(v_3)$.
- If the last line is "Uncolor, event 2", then $M[i] = \varphi_i(v_1)$.
- If the last line is "Uncolor, event 3", then $M[i] = \varphi_i(v_1)$.
- If the last line is "Uncolor, event 4", then $M[i] = \varphi_i(u)$.
- If the last line is "Uncolor, event 5", then $M[i] = \varphi_i(v_2)$.
- If the last line is "Uncolor, event 6", then $M[i] = \varphi_i(v_2)$.
- If the last line is "Uncolor, event 7", then $M[i] = \varphi_i(v_3)$.
- If the last line is "Uncolor, event 8", then $M[i] = \varphi_i(u)$.
- If the last line is "Uncolor, event 9", then $M[i] = \varphi_i(v_3)$.
- If the last line is "Uncolor, event 10", then $M[i] = \varphi_i(u)$.
- If the last line is "Uncolor, event 11", then $M[i] = \varphi_i(v_2)$.
- If the last line is "Uncolor, event 12", then $M[i] = \varphi_i(v_2)$.

Therefore, one can recover M_i from (φ_i, R_i). This concludes the proof of the Lemma 3.1. □

Let us now bound the number of possible records.

Lemma 3.2 *Algorithm DegeneratestarColoring_G produces at most* $o(\kappa^t)$ *distinct records* R.

Proof Since algorithm DegeneratestarColoring fails to color G, the record R has exactly t "`Color`" lines (i.e. the algorithm consumes the whole input vector). It contains also "`Uncolor`" lines of different events: "`Event 1-12`", Let $\mathcal{T} = \{1, 2, \ldots, 12\}$ be the set of events. Let us denote by s_j the number of uncolored vertices when a event j occurs. Observe that:

- For every "`Uncolor, event 1`" step, the algorithm DegeneratestarColoring uncolors 6 previously colored vertex. Hence set $s_1 = 6$.

- For every "`Uncolor, event 2`" step, the algorithm DegeneratestarColoring uncolors 2 previously colored vertices. Hence set $s_2 = 2$.
- For every "`Uncolor, event 3`" step, the algorithm DegeneratestarColoring uncolors 2 previously colored vertices. Hence set $s_3 = 2$.
- For every "`Uncolor, event 4`" step, the algorithm DegeneratestarColoring uncolors 4 previously colored vertices. Hence set $s_4 = 4$.
- For every "`Uncolor, event 5`" step, the algorithm DegeneratestarColoring uncolors 4 previously colored vertices. Hence set $s_5 = 4$.
- For every "`Uncolor, event 6`" step, the algorithm DegeneratestarColoring uncolors 4 previously colored vertices. Hence set $s_6 = 4$.
- For every "`Uncolor, event 7`" step, the algorithm DegeneratestarColoring uncolors 6 previously colored vertices. Hence set $s_7 = 6$.
- For every "`Uncolor, event 8`" step, the algorithm DegeneratestarColoring uncolors 1 previously colored vertices. Hence set $s_8 = 1$.
- For every "`Uncolor, event 9`" step, the algorithm DegeneratestarColoring uncolors 5 previously colored vertices. Hence set $s_9 = 5$.
- For every "`Uncolor, event 10`" step, the algorithm DegeneratestarColoring uncolors 1 previously colored vertices. Hence set $s_{10} = 1$.
- For every "`Uncolor, event 11`" step, the algorithm DegeneratestarColoring uncolors 6 previously colored vertices. Hence set $s_{11} = 6$.
- For every "`Uncolor, event 12`" step, the algorithm DegeneratestarColoring uncolors 4 previously colored vertices. Hence set $s_{12} = 4$.

To compute the total number of possible records, let us compute how many different entries, denoted by C_j, an "`Uncolor`" step of event j can produce in the record. Observe that:

- An "`Uncolor, event 1`" line can produce at most $\frac{3}{4}\Delta^5$ different entries in the record, according to Fig. 1(1), set $C_1 = \frac{3}{4}\Delta^5$.
- An "`Uncolor, event 2`"line can produce at most Δ^3 different entries in the record, according to Fig. 1(2), set $C_2 = \Delta^3$.
- An "`Uncolor, event 3`"line can produce at most $\frac{\Delta^3}{6}$ different entries in the record, according to Fig. 1(3), set $C_3 = \frac{\Delta^3}{6}$.
- An "`Uncolor, event 4`"line can produce at most $\frac{\Delta^5+\Delta^4+2\Delta^3}{4}$ different entries in the record, according to Fig. 1(4), set $C_4 = \frac{\Delta^5+\Delta^4+2\Delta^3}{4}$.
- An "`Uncolor, event 5`"line can produce at most $\frac{3}{4}\Delta^4$ different entries in the record, according to Fig. 1(5), set $C_5 = \frac{3}{4}\Delta^4$.
- An "`Uncolor, event 6`"line can produce at most $\frac{3}{4}\Delta^4$ different entries in the record, according to Fig. 1(6), set $C_6 = \frac{3}{4}\Delta^4$.
- An "`Uncolor, event 7`"line can produce at most $\frac{\Delta^6+2\Delta^5+4\Delta^4}{8}$ different entries in the record, according to Fig. 1(7), set $C_7 = \frac{\Delta^6+2\Delta^5+4\Delta^4}{8}$.
- An "`Uncolor, event 8`"line can produce at most $\Delta^{\frac{3}{2}}$ different entries in the record, according to Fig. 1(8), set $C_8 = \Delta^{\frac{3}{2}}$.

- An "`Uncolor, event 9`"line can produce at most $\frac{\Delta^5}{2}$ different entries in the record, according to Fig. 1(9), set $C_9 = \frac{\Delta^5}{2}$.
- An "`Uncolor, event 10`"line can produce at most Δ different entries in the record, according to Fig. 1(10), set $C_{10} = \Delta$.
- An "`Uncolor, event 11`"line can produce at most $\frac{\Delta^6+2\Delta^5+2\Delta^4}{8}$ different entries in the record, according to Fig. 1(11), set $C_{11} = \frac{\Delta^6+2\Delta^5+2\Delta^4}{8}$.
- An "`Uncolor, event 12`"line can produce at most $\frac{\Delta^4}{2}$ different entries in the record, according to Fig. 1(12), set $C_{12} = \frac{\Delta^4}{2}$.

Now we proceed to compute the total number of different records. Recall that the record R has exactly t "Color" steps and some "Uncolor" steps of different events. Let $t_i (i = 1, 2, \ldots, 12)$ be the number of "Uncolor" steps of events "event 1", "event 2", ..., "event 12", respectively. From the above discussion we know that the number of each "Uncolor" step "event i" uncolor s_i previously colored vertex, thus $\sum_{1\le i\le 12} s_i t_i$ equals to the number of uncolored vertices during the execution of algorithm DegeneratestarColoring, so $\sum_{1\le i\le 12} s_i t_i \le t$. From the hypothesis we know that at the end of the execution of algorithm DegeneratestarColoring there are less than n colored vertices. Therefore,

$$t - n < \sum_{1\le i\le 12} s_i t_i \le t. \tag{1}$$

Based on (1) and the discussions above, we begin to count the number $\#Seq(t_1, t_2 \ldots, t_{12})$ of possible sequences of "Color", "Uncolor, event 1","Uncolor, event 2", ... ,"Uncolor, event 12" steps in the record, for the fixed $t_1, t_2, \ldots, t_{12}$. In the following, let $t_0 = t - \sum_{1\le i\le 12} t_i$. From algorithm DegeneratestarColoring we can obtain

$$\begin{aligned}\#Seq(t_1, t_2 \ldots, t_{12}) &\le \binom{t}{t_0} \times \binom{t - t_0}{t_1} \times \cdots \times \binom{t - \sum_{0\le i\le 10} t_i}{t_{11}} \\ &\le \binom{t}{t_0, t_1, \ldots, t_{12}}.\end{aligned}$$

The computation of C_i and $\#Seq(t_1, t_2 \ldots, t_{12})$ implies that for fixed $t, t_0, \ldots, t_{12}$, the number of different records is bounded by the following function B_t:

$$B_t(t_0, \ldots, t_{12}) = \binom{t}{t_0, t_1, \ldots, t_{12}} \times \prod_{1\le i\le 12} C_i^{t_i}.$$

Sum over all possible 13−tuples $t_0, \ldots, t_{12}$ satisfying Eq. (1), we can bound the number of different records $\#REC$ by following inequality:

$$\#REC \le \sum_{t_0,\ldots,t_{12}} B_t(t_0, \ldots, t_{12}).$$

From Eq. (1), we can obtain the following equations

$$\sum_{t_0,\ldots,t_{12}} t_i = t$$

and

$$t \geq \sum_{1\leq i\leq 12} s_i t_i. \tag{2}$$

In order to complete the proof of Lemma 3.2, we present the following Lemma given by Goncalves et al. (2014).

Lemma 3.3 (Goncalves et al. 2014) *Summing over all possible $(p+1)$-tuples $t_0, \ldots, t_p$ satisfying Eq. (2), we have for sufficiently large t that*

$$\sum_{t_0,\ldots,t_p} B_t(t_0,\ldots,t_p) < t(t+1)^p (\inf_{0<x\leq 1} Q(x))^t.$$

By Lemma 3.3, we know that for the sufficiently large t,

$$\#REC < t(t+1)^{12} (\inf_{0<x\leq 1} Q(x))^t,$$

for $Q(x) = \frac{1}{x}(1 + \sum_{1\leq i\leq 12}(C_i x^{s_i}))$ with C_i and s_i given as above (the s_i's satisfy Eq. (2) by Eq. (1)) and any real $0 < x \leq 1$. Hence we have

$$Q(x) = \frac{1}{x}(1 + C_1x^6 + C_2x^2 + C_3x^2 + C_4x^4 + C_5x^4 + C_6x^4 + C_7x^6 + C_8x + C_9x^5 + C_{10}x + C_{11}x^6 + C_{12}x^4).$$

Then we substitute the value for respective C_i, $i = 1, 2, \ldots, 12$.
Setting $X = \frac{1}{\Delta^{\frac{3}{2}}}$, we have

$$Q(X) < \frac{19}{6}\Delta^{\frac{3}{2}} + 5\Delta \leq \kappa.$$

Finally, we have $\#REC = o(\kappa^t)$. By Lemma 3.1, we can obtain contradiction, thus completes the proof of Theorem 4. □

Remark 1 Obviously, using Entropy Compression Method, we can easily extend the result of Theorem 4 to its list version.

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