Oriented Bicyclic Graphs with the First Five Large Skew Energies*

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Abstract

Let G^σ be an oriented graph obtained by assigning an orientation σ to the edge set of a simple undirected graph G. Let $S(G^\sigma)$ be the skew adjacency matrix of G^σ. The skew energy of G^σ is defined as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$. In this paper, we give the skew energy order of a family of digraphs and determine the oriented bicyclic graphs of order $n \geq 13$ with the first five largest skew energies, which extends the results of the paper [X. Shen, Y. Hou, C. Zhang, Bicyclic digraphs with extremal skew energy, Electron. J. Linear Algebra 23 (2012) 340–355].

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1 Introduction

An important quantum-chemical characteristic of a conjugated molecule is its total $\pi-$electron energy. The energy of a graph has closed links to chemistry. Let G be a simple undirected graph and $A(G)$ be the adjacency matrix of G. Gutman [7] firstly defined the *energy* $E(G)$ of G as follows:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

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where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$. For more results about graph energy, we refer the readers to the surveys [8, 9], the book [14] and the recent papers [16, 18].

There are various generalizations of graph energy, such as the Randić energy [5, 15], the distance energy [22], the incidence energy [2, 3] and the energy of a polynomial [11, 17]. In this paper, we focus on the *skew energy* of a graph. Let G^σ be an oriented graph obtained by assigning an orientation σ to the edge set of a simple undirected graph G. The *skew adjacency matrix* $S(G^\sigma) = (s_{ij})$ of G^σ is a real skew symmetric matrix, where $s_{ij} = 1$ and $s_{ji} = -1$ if ij is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. Then the authors [1] defined the *skew energy* $E_S(G^\sigma)$ of an oriented graph G^σ as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$. The *skew characteristic polynomial* of G^σ is defined as

$$P_S(G^\sigma; x) = \det(xI - S(G^\sigma)) = \sum_{i=0}^{n} b_i x^{n-i}.$$

Since $S(G^\sigma)$ is a real skew symmetric matrix, we have $b_{2k}(G^\sigma) \geq 0$ and $b_{2k+1}(G^\sigma) = 0$ for all $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ (see [6]). Thus we have

$$P_S(G^\sigma; x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{2k}(G^\sigma) x^{n-2k}.$$

By the coefficients of $P_S(G^\sigma; x)$, the skew energy $\mathcal{E}_S(G^\sigma)$ can be expressed by the following integral formula as follows [13]:

$$\mathcal{E}_S(G^\sigma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{t^2} \ln(1 + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{2k} t^{2k}) \mathrm{d}t.$$

Thus $\mathcal{E}_S(G^\sigma)$ is a strictly monotonically increasing function of $b_{2k}(G^\sigma), k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Consequently, if G^{σ_1} and H^{σ_2} are oriented graphs with

$$b_{2k}(G^{\sigma_1}) \geq b_{2k}(H^{\sigma_2}) \quad \text{for each } k \ (0 \leq k \leq \lfloor \frac{n}{2} \rfloor), \tag{1}$$

then

$$\mathcal{E}_S(G^{\sigma_1}) \geq \mathcal{E}_S(H^{\sigma_2}). \tag{2}$$

Equality in (2) is attained only if (1) is an equality for all $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. If the inequalities (1) hold for all k, then we write $G \succeq H$ or $H \preceq G$. If $G \succeq H$, but not $H \succeq G$, then we write $G \succ H$. That is exactly the *quasi-order relation* defined by Gutman and Polansky [10] on graph energy, which is generalized to the skew-energy of oriented graph. See

[6, 13, 19, 20, 23, 25] for some recent results about the spectrum and energy of the skew-adjacency matrix.

Due to the coefficients $b_{2k} \geq 0$, it makes that the skew energy problem is much easier than the adjacency energy problems. Particularly, as far as the unicyclic and bicyclic graphs are concerned, Hou et al. [13] determined the oriented unicyclic graphs with the minimum and the maximum skew energy respectively. Subsequently, they also [19] characterized the bicyclic digraphs with the minimum and the maximum skew energy respectively. Very recently, Wang et al. [24] identified the bicyclic digraphs with the second maximum skew energy. In this paper, we will determine the oriented bicyclic graphs of order $n \geq 13$ with the first five largest skew energies, which extends the results in [19, 24].

For the sake of completeness, we say something about the orientation of G^σ that already exists [19]. Let G^σ be an orientation of a graph G. If C is an even cycle of G, then we say C is *evenly oriented* relative to G^σ if it has an even number of edges oriented in the direction of the routing; otherwise C is *oddly oriented*. Let W be a subset of $V(G)$ and $\overline{W} = V(G)\backslash W$. The orientation $G^{\sigma'}$ of G obtained from G^σ by reversing the orientations of all arcs between $\overline{W}$ and W is said to be obtained from G^σ by a switching with respect to W. Moreover, two orientations G^σ and $G^{\sigma'}$ of a graph G are said to be switching-equivalent if $G^{\sigma'}$ can be obtained from G^σ by a sequence of switchings. As noted in [1], since the skew adjacency matrices obtained by a switching are similar, their spectra and hence skew energies are equal.

It is easy to verify that up to switching equivalence there are just two orientations of a cycle C: (1) Just one edge on the cycle has the opposite orientation to that of others, we call it orientation $+$. (2) All edges on the cycle C have the same orientation, we denote this orientation $-$. So if a cycle is of even length and oddly oriented, then it is equivalent to the orientation $+$; if a cycle is of even length and evenly oriented, then it is equivalent to the orientation $-$. The skew energy of a directed tree is the same as the energy of its underlying tree ([1]). So by switching equivalence, for an oriented unicyclic graph or an oriented bicyclic graph, we only need to consider the orientations of cycles. Simultaneously, we denote by T the oriented tree and omit the superscript σ since the skew energy of a directed tree is independent of its orientations.

We denote by G^+ (resp., G^-) the unicyclic graph on which the orientation of a cycle is of orientation $+$ (resp., $-$), and denote by G^* the unicyclic graph on which the orientation of a cycle is of arbitrary orientation $*$. Let C_x, C_y be two cycles in bicyclic graph G with t ($t \geq 0$) common vertices. If $t \leq 1$, then G contains exactly two cycles, and we denote by $G^{a,b}$ the bicyclic graph on which cycle C_x is of orientation a and cycle C_y is of orientation b, where $a, b \in \{+, -, *\}$. If $t \geq 2$, then G contains exactly three cycles. The third cycle is denoted by C_z, where $z = x+y-2t+2$. Without loss of generality, assume that $x \leq z$ and $y \leq z$. Moreover, Let $G^{a,b,c}$ be the bicyclic graph on which cycle C_x is of orientation a, cycle C_y is of orientation b, C_z is of orientation c, where $a, b, c \in \{+, -, *\}$. The other graphs used in this paper are shown in Fig. 1.

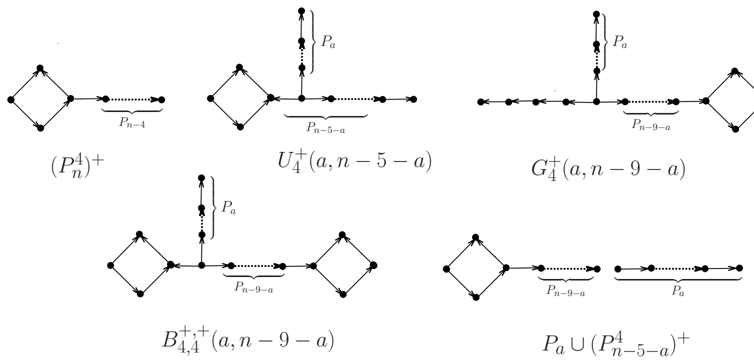

Figure 1: Graphs used in the paper.

The rest of this paper is organized as follows. In section 2, some useful lemmas are stated. In section 3, the quasi-order relations of some graphs are discussed. In section 4, the oriented bicyclic graphs of order $n \geq 13$ with the first five largest skew energies are determined.

2 Some useful lemmas

Let G be a graph. A *linear subgraph* L of G is a disjoint union of some edges and some cycles in G [4]. We call a linear subgraph L of G *evenly linear* if L contains no cycle with odd length and denote by $\mathcal{EL}_i(G)$ the set of all evenly linear subgraphs of G with i vertices. For a linear subgraph $L \in \mathcal{EL}_i(G)$, denote by $p_e(L)$ (resp., $p_o(L)$) the number of evenly (resp., oddly) oriented cycles in L relative to G^σ.

Lemma 2.1 *[12] Let G^σ be an orientation of a graph G. Then*

$$b_i(G^\sigma) = \sum_{L \in \mathcal{EL}_i} (-2)^{p_e(L)} 2^{p_o(L)}.$$

Lemma 2.1 implies that $b_{2k}(G^\sigma) = m(G^\sigma, k)$ for any orientation of a graph that does not contain any even cycle, particularly for a tree or a unicyclic non-bipartite graph.

Lemma 2.2 *[12] Let $e = uv$ be an edge of G. Then*

$$P_S(G^\sigma;x) = P_S(G^\sigma - e; x) + P_S(G^\sigma - u - v; x) + 2\sum_{e\in C\in Od(G^\sigma)} P_S(G^\sigma - C; x) - 2\sum_{e\in C\in Ev(G^\sigma)} P_S(G^\sigma - C; x).$$

Corollary 2.1 *[12] Let $e = uv$ be an edge of G that is on no even cycle of G. Then*

$$P_S(G^\sigma; x) = P_S(G^\sigma - e; x) + P_S(G^\sigma - u - v; x). \quad (3)$$

By equating the coefficient of polynomials in Eq.(3), we have

$$b_{2k}(G^\sigma) = b_{2k}(G^\sigma - e) + b_{2k-2}(G^\sigma - u - v). \quad (4)$$

Furthermore, if $e = uv$ is a pendant edge with pendant vertex v, then

$$b_{2k}(G^\sigma) = b_{2k}(G^\sigma - v) + b_{2k-2}(G^\sigma - u - v). \quad (5)$$

A *k-matching* M of a graph G is a disjoint union of k-edges. The number of k-matchings of G is denoted by $m(G, k)$.

Lemma 2.3 *[13] Let $e = uv$ be an edge of G of order n. Then*
(1) $m(G, k) = m(G - e, k) + m(G - u - v, k - 1)$.
(2) If G is a forest, then $m(G, k) \le m(P_n, k)$, $k \ge 1$.
(3) If H is a subgraph of G, then $m(H, k) \le m(G, k)$, $k \ge 1$. Moreover, if H is a proper subgraph of G, then the inequality is strict.
We define $m(G, 0) = 1$ and $m(G, k) = 0$ for $k \ge \frac{n}{2}$.

Lemma 2.4 *[21] Let $a + b = c + d$ with $0 \le a \le b$ and $0 \le c \le d$. Let $a < c$. Then*
(1) if a is even, then $m(P_a \cup P_b, i) \ge m(P_c \cup P_d, i)$. Furthermore, there exists at least one index i such that the above inequality is strict.
(2) if a is odd, then $m(P_a \cup P_b, i) \le m(P_c \cup P_d, i)$. Furthermore, there exists at least one index i such that the above inequality is strict.

Two results are immediately followed from Lemma 2.3 and 2.4.

Lemma 2.5 *[19] Let F_n be a (oriented) forest of order n. Then $F_n \preceq P_n$. Equality holds if and only if $F_n = P_n$.*

Lemma 2.6 *[19] $P_n \succ P_2 \cup P_{n-2} \succ P_4 \cup P_{n-4} \succ \cdots P_{2k} \cup P_{n-2k} \succ P_{2k+1} \cup P_{n-2k-1} \succ P_{2k-1} \cup P_{n-2k+1} \succ \cdots \succ P_3 \cup P_{n-3} \succ P_1 \cup P_{n-1}$.*

Let $\mathcal{B}_n^+ = \{U_4^+(a, b) | 0 \le a \le b,\ a + b = n - 5\}$.

Lemma 2.7 *[25] Let $k = \lfloor \frac{n-5}{2} \rfloor$, $t = \lfloor \frac{k}{2} \rfloor$ and $\ell = \lfloor \frac{k-1}{2} \rfloor$. Then we have the following quasi-order relation in $\mathcal{B}_n^+$:*
$U_4^+(0, n-5) \succ U_4^+(2, n-7) \succ \cdots \succ U_4^+(2t, n-5-2t) \succ U_4^+(2\ell+1, n-5-2\ell-1) \succ \cdots \succ U_4^+(7, n-12) \succ U_4^+(5, n-10) \succ U_4^+(3, n-8) \succ U_4^+(1, n-6)$.

Let $\mathcal{A}_n^+ = \mathcal{B}_n^+ \backslash \{U_4^+(5, n-10),\ U_4^+(3, n-8),\ U_4^+(1, n-6)\}$.

Lemma 2.8 *[25] Let $n \geq 31$. The oriented unicyclic graphs of order n with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies are the oriented unicyclic graphs in $\mathcal{A}_n^+$.*

3 The Quasi-order Relation in $\mathcal{C}_n^+$

Let $\mathcal{C}_n^+ = \{B_{4,4}^{+,+}(a, n-9-a) | 0 \leq a \leq n-9\}$ and $\mathcal{D}_{n-5}^+ = \{P_a \cup (P_{n-5-a}^4)^+ | 0 \leq a \leq n-9\}$. In this section, we determine the quasi-order relation in $\mathcal{D}_{n-5}^+$ for $n \geq 13$, and then apply it to obtain the quasi-order relation in $\mathcal{C}_n^+$ for $n \geq 13$.

Lemma 3.1 *Let $0 \leq a \leq \lfloor \frac{n-10}{2} \rfloor$.*
(1) If a is even, then $P_a \cup (P_{n-5-a}^4)^+ \succ P_{n-9-a} \cup (P_{a+4}^4)^+$.
(2) If a is odd, then $P_a \cup (P_{n-5-a}^4)^+ \prec P_{n-9-a} \cup (P_{a+4}^4)^+$.

Proof. The conditions of the lemma shows that $a < n-9-a$. Let $e_1 = u_1v_1$ be the edge of $P_a \cup (P_{n-5-a}^4)^+$ which connects the cycle C_4^+ and the path P_{n-9-a}, and $e_2 = u_2v_2$ be the edge of $P_{n-9-a} \cup (P_{a+4}^4)^+$ which connects C_4^+ and P_a. By Lemma 2.2, we get

$$b_{2k}(P_a \cup (P_{n-5-a}^4)^+) = b_{2k}(C_4^+ \cup P_a \cup P_{n-9-a}) + b_{2k-2}(P_3 \cup P_a \cup P_{n-10-a}),$$

$$b_{2k}(P_{n-9-a} \cup (P_{a+4}^4)^+) = b_{2k}(C_4^+ \cup P_a \cup P_{n-9-a}) + b_{2k-2}(P_3 \cup P_{a-1} \cup P_{n-9-a}).$$

(1) If a is even and $a < n-9-a$, then $a-1$ is odd and $a-1 \leq n-9-a$. By Lemma 2.6 we have that

$$P_a \cup P_{n-10-a} \succ P_{a-1} \cup P_{n-9-a}.$$

Then $b_{2k}(P_a \cup (P_{n-5-a}^4)^+) \geq b_{2k}(P_{n-9-a} \cup (P_{a+4}^4)^+)$ and there exists at least one index k such that the above inequality is strict. Hence, $P_a \cup (P_{n-5-a}^4)^+ \succ P_{n-9-a} \cup (P_{a+4}^4)^+$.

(2) If a is odd and $a < n-9-a$, then $a-1$ is even and $a-1 \leq n-9-a$. By Lemma 2.6 we have that

$$P_a \cup P_{n-10-a} \prec P_{a-1} \cup P_{n-9-a}.$$

Then $b_{2k}(P_a \cup (P^4_{n-5-a})^+) \le b_{2k}(P_{n-9-a} \cup (P^4_{a+4})^+)$ and there exists at least one index k such that the above inequality is strict. Hence $P_a \cup (P^4_{n-5-a})^+ \prec P_{n-9-a} \cup (P^4_{a+4})^+$.

This completes the proof. □

Lemma 3.2 *Let $a < b \le \lfloor \frac{n-10}{2} \rfloor$.*

(1) If a is even, then $P_a \cup (P^4_{n-5-a})^+ \succ P_b \cup (P^4_{n-5-b})^+$.

(2) If a is odd, then $P_a \cup (P^4_{n-5-a})^+ \prec P_b \cup (P^4_{n-5-b})^+$.

Proof. If $a < b \le \lfloor \frac{n-10}{2} \rfloor$, then $a \le n-9-a$, $a \le n-10-a$, $b \le n-9-b$ and $b \le n-10-b$. Let $e_1 = u_1v_1$ be the edge of $P_a \cup (P^4_{n-5-a})^+$ which connects the cycle C_4^+ and the path P_{n-9-a}, and $e_2 = u_2v_2$ be the edge of $P_b \cup (P^4_{n-5-b})^+$ which connects C_4^+ and P_{n-9-b}. By Lemma 2.2, we get

$$b_{2k}(P_a \cup (P^4_{n-5-a})^+) = b_{2k}(C_4^+ \cup P_a \cup P_{n-9-a}) + b_{2k-2}(P_3 \cup P_a \cup P_{n-10-a}),$$

$$b_{2k}(P_b \cup (P^4_{n-5-b})^+) = b_{2k}(C_4^+ \cup P_b \cup P_{n-9-b}) + b_{2k-2}(P_3 \cup P_b \cup P_{n-10-b}).$$

(1) If a is even and $a < b$, by Lemma 2.6 we have that

$$P_a \cup P_{n-9-a} \succ P_b \cup P_{n-9-b},\ P_a \cup P_{n-10-a} \succ P_b \cup P_{n-10-b}.$$

Then $b_{2k}(P_a \cup (P^4_{n-5-a})^+) \ge b_{2k}(P_b \cup (P^4_{n-5-b})^+)$ and there exists at least one index k such that the above inequality is strict. Hence, $P_a \cup (P^4_{n-5-a})^+ \succ P_b \cup (P^4_{n-5-b})^+$.

(2) If a is odd and $a < b$, by Lemma 2.6 we have that

$$P_a \cup P_{n-9-a} \prec P_b \cup P_{n-9-b},\ P_a \cup P_{n-10-a} \prec P_b \cup P_{n-10-b}.$$

Then $b_{2k}(P_a \cup (P^4_{n-5-a})^+) \le b_{2k}(P_b \cup (P^4_{n-5-b})^+)$ and there exists at least one index k such that the above inequality is strict. Hence, $P_a \cup (P^4_{n-5-a})^+ \prec P_b \cup (P^4_{n-5-b})^+$.

This finishes the proof. □

Lemma 3.3 *Let $\lceil \frac{n-5}{2} \rceil \le b < a$ and $a' = n-9-a$.*

(1) If a' is even, then $P_a \cup (P^4_{n-5-a})^+ \succ P_b \cup (P^4_{n-5-b})^+$.

(2) If a' is odd, then $P_a \cup (P^4_{n-5-a})^+ \prec P_b \cup (P^4_{n-5-b})^+$.

Proof. Set $b' = n-9-b$. Then $a' < b' \le \lfloor \frac{n-13}{2} \rfloor$. Let $e_1 = u_1v_1$ be the edge on the cycle C_4^+ of $P_a \cup (P^4_{n-5-a})^+$ such that u_1 is the vertex on the path, and $e_2 = u_2v_2$ be the edge on the cycle C_4^+ of $P_b \cup (P^4_{n-5-b})^+$ such that u_2 is the vertex on the path. By Lemma 2.2, we get

$$b_{2k}(P_a \cup (P^4_{n-5-a})^+) = b_{2k}((P^4_{a'+4})^+ \cup P_{n-9-a'})$$

$$=b_{2k}(P_{a'+4}\cup P_{n-9-a'})+b_{2k-2}(P_2\cup P_{a'}\cup P_{n-9-a'})$$
$$+2b_{2k-4}(P_{a'}\cup P_{n-9-a'}),$$

and

$$b_{2k}(P_b\cup (P^4_{n-5-b})^+)=b_{2k}((P^4_{b'+4})^+\cup P_{n-9-b'})$$
$$=b_{2k}(P_{b'+4}\cup P_{n-9-b'})+b_{2k-2}(P_2\cup P_{b'}\cup P_{n-9-b'})$$
$$+2b_{2k-4}(P_{b'}\cup P_{n-9-b'}).$$

(1) If a' is even and $a'<b'\leq\lfloor\frac{n-13}{2}\rfloor$, by Lemma 2.6 we have that

$$P_{a'+4}\cup P_{n-9-a'}\succ P_{b'+4}\cup P_{n-9-b'},\ P_{a'}\cup P_{n-9-a'}\succ P_{b'}\cup P_{n-9-b'},$$

then we get $b_{2k}((P^4_{a'+4})^+\cup P_{n-9-a'})\geq b_{2k}((P^4_{b'+4})^+\cup P_{n-9-b'})$ and there exists at least one index k such that the above inequality is strict. Thus, $b_{2k}(P_a\cup (P^4_{n-5-a})^+)\geq b_{2k}(P_b\cup (P^4_{n-5-b})^+)$ and there exists at least one index k such that the above inequality is strict. Hence, $P_a\cup (P^4_{n-5-a})^+\succ P_b\cup (P^4_{n-5-b})^+$.

(2) If a' is odd and $a'<b'\leq\lfloor\frac{n-13}{2}\rfloor$, by Lemma 2.6 we have that

$$P_{a'+4}\cup P_{n-9-a'}\prec P_{b'+4}\cup P_{n-9-b'},\ P_{a'}\cup P_{n-9-a'}\prec P_{b'}\cup P_{n-9-b'}.$$

Then we get $b_{2k}((P^4_{a'+4})^+\cup P_{n-9-a'})\leq b_{2k}((P^4_{b'+4})^+\cup P_{n-9-b'})$ and there exists at least one index k such that the above inequality is strict. Thus, $b_{2k}(P_a\cup (P^4_{n-5-a})^+)\leq b_{2k}(P_b\cup (P^4_{n-5-b})^+)$ and there exists at least one index k such that the above inequality is strict. Hence, $P_a\cup (P^4_{n-5-a})^+\prec P_b\cup (P^4_{n-5-b})^+$. $\square$

With the similar techniques to those of Lemma 3.2 and 3.3, it is easy to obtain the following results by Lemma 2.4.

Lemma 3.4 *(1)* $n\equiv 1(\mathrm{mod}4)$, $P_{\frac{n-5}{2}}\cup (P^4_{\frac{n-5}{2}})^+\succ P_{\frac{n-9}{2}}\cup (P^4_{\frac{n-1}{2}})^+\succ P_{\frac{n-7}{2}}\cup (P^4_{\frac{n-3}{2}})^+\succ P_{\frac{n-3}{2}}\cup (P^4_{\frac{n-7}{2}})^+$.

(2) $n\equiv 3(\mathrm{mod}4)$, $P_{\frac{n-3}{2}}\cup (P^4_{\frac{n-7}{2}})^+\succ P_{\frac{n-7}{2}}\cup (P^4_{\frac{n-3}{2}})^+\succ P_{\frac{n-9}{2}}\cup (P^4_{\frac{n-1}{2}})^+\succ P_{\frac{n-5}{2}}\cup (P^4_{\frac{n-5}{2}})^+$.

(3) $n\equiv 2(\mathrm{mod}4)$, $P_{\frac{n-4}{2}}\cup (P^4_{\frac{n-6}{2}})^+\succ P_{\frac{n-8}{2}}\cup (P^4_{\frac{n-2}{2}})^+\succ P_{\frac{n-6}{2}}\cup (P^4_{\frac{n-4}{2}})^+\succ P_{\frac{n-2}{2}}\cup (P^4_{\frac{n-8}{2}})^+$.

(4) $n\equiv 0(\mathrm{mod}4)$, $P_{\frac{n-2}{2}}\cup (P^4_{\frac{n-8}{2}})^+\succ P_{\frac{n-6}{2}}\cup (P^4_{\frac{n-4}{2}})^+\succ P_{\frac{n-8}{2}}\cup (P^4_{\frac{n-2}{2}})^+\succ P_{\frac{n-2}{2}}\cup (P^4_{\frac{n-8}{2}})^+$.

Lemma 3.5 *(1)* $P_{n-9}\cup (P^4_4)^+\succ P_2\cup (P^4_{n-7})^+$.

(2) If a is even and $2 \le a \le \lfloor \frac{n-12}{2} \rfloor$, then $P_{n-9-a} \cup (P_{a+4}^4)^+ \succ P_{a+2} \cup (P_{n-7-a}^4)^+$.

(3) If a is odd and $1 \le a \le \lfloor \frac{n-12}{2} \rfloor$, then $P_{n-9-a} \cup (P_{a+4}^4)^+ \prec P_{a+2} \cup (P_{n-7-a}^4)^+$.

Proof. (1) We can choose the edge $e_1 = u_1v_1$ on the path P_{n-9} of $P_{n-9} \cup (P_4^4)^+$ which connects P_2 and P_{n-11}, and the edge $e_2 = u_2v_2$ on the path of the uncyclic graph $(P_{n-7}^4)^+$ of $P_2 \cup (P_{n-7}^4)^+$ which connects C_4 and P_{n-11}. By Lemma 2.2 we get

$$b_{2k}(P_{n-9} \cup (p_4^4)^+) = b_{2k}(C_4^+ \cup P_2 \cup P_{n-11}) + b_{2k-2}(C_4^+ \cup P_1 \cup P_{n-12}),$$

and

$$b_{2k}(P_2 \cup (p_{n-7}^4)^+) = b_{2k}(C_4^+ \cup P_2 \cup P_{n-11}) + b_{2k-2}(P_3 \cup P_2 \cup P_{n-12}).$$

Easily to verify that $C_4^+ \cup P_1 \succ P_3 \cup P_2$. So, $b_{2k-2}(C_4^+ \cup P_1 \cup P_{n-12}) > b_{2k-2}(P_3 \cup P_2 \cup P_{n-12})$. Then, $b_{2k}(P_{n-9} \cup (p_4^4)^+) \ge b_{2k}(P_2 \cup (p_{n-7}^4)^+)$ and there exists at least one index k such that the above inequality is strict. Hence $P_{n-9} \cup (P_4^4)^+ \succ P_2 \cup (P_{n-7}^4)^+$.

(2) By Lemma 2.1 we have that

$$\begin{aligned}
&b_{2k}(P_{n-9-a} \cup (P_{a+4}^4)^+) \\
=&m(P_{a+4}^4 \cup P_{n-9-a}, k) + 2m(P_a \cup P_{n-9-a}, k-2) \\
=&m(P_{a+4} \cup P_{n-9-a}, k) + m(P_a \cup P_{n-9-a}, k-1) \\
&+ 3m(P_a \cup P_{n-9-a}, k-2),
\end{aligned}$$

and

$$\begin{aligned}
&b_{2k}(P_{a+2} \cup (P_{n-7-a}^4)^+) \\
=&m(P_{n-7-a}^4 \cup P_{a+2}, k) + 2m(P_{a+2} \cup P_{n-11-a}, k-2) \\
=&m(P_{a+2} \cup P_{n-7-a}, k) + m(P_{a+2} \cup P_{n-11-a}, k-1) \\
&+ 3m(P_{a+2} \cup P_{n-11-a}, k-2).
\end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned}
&b_{2k}(P_{n-9-a} \cup (P_{a+4}^4)^+) - b_{2k}(P_{a+2} \cup (P_{n-7-a}^4)^+) \\
=&m(P_{a+4} \cup P_{n-9-a}, k) + m(P_a \cup P_{n-9-a}, k-1) \\
&+ 3m(P_a \cup P_{n-9-a}, k-2) - m(P_{a+2} \cup P_{n-7-a}, k) \\
&- m(P_{a+2} \cup P_{n-11-a}, k-1) - 3m(P_{a+2} \cup P_{n-11-a}, k-2) \\
=&m(P_{a+1} \cup P_{n-9-a}, k-1) - m(P_{a+2} \cup P_{n-10-a}, k-1)
\end{aligned}$$

$$
\begin{aligned}
&+ m(P_a \cup P_{n-9-a}, k-1) + 3m(P_a \cup P_{n-9-a}, k-2)\\
&- m(P_{a+2} \cup P_{n-11-a}, k-1) - 3m(P_{a+2} \cup P_{n-11-a}, k-2)\\
=&m(P_{a+1} \cup P_{n-10-a}, k-1) + m(P_a \cup P_{n-11-a}, k-2)\\
&+ m(P_{a-1} \cup P_{n-11-a}, k-3) - m(P_{a+1} \cup P_{n-10-a}, k-1)\\
&- m(P_a \cup P_{n-11-a}, k-2) - m(P_a \cup P_{n-12-a}, k-3)\\
&+ m(P_a \cup P_{n-11-a} \cup P_2, k-1) + m(P_a \cup P_{n-12-a}, k-2)\\
&+ 3m(P_a \cup P_{n-11-a} \cup P_2, k-2) + 3m(P_a \cup P_{n-12-a}, k-3)\\
&- m(P_a \cup P_2 \cup P_{n-11-a}, k-1) - m(P_{a-1} \cup P_{n-11-a}, k-2)\\
&- 3m(P_a \cup P_2 \cup P_{n-11-a}, k-2) - 3m(P_{a-1} \cup P_{n-11-a}, k-3)\\
=&2m(P_a \cup P_{n-12-a}, k-3) - 2m(P_{a-1} \cup P_{n-11-a}, k-3)\\
&+ m(P_a \cup P_{n-12-a}, k-2) - m(P_{a-1} \cup P_{n-11-a}, k-2).
\end{aligned}
$$

If a is even and $2 \leq a \leq \lfloor \frac{n-12}{2} \rfloor$, then by Lemma 2.4 we have

$$m(P_a \cup P_{n-12-a}, k-2) \geq m(P_{a-1} \cup P_{n-11-a}, k-2),$$

and

$$m(P_a \cup P_{n-12-a}, k-3) \geq m(P_{a-1} \cup P_{n-11-a}, k-3).$$

Furthermore, there exists at least one index k such that the above inequality is strict.

So, we have $b_{2k}(P_{n-9-a} \cup (P^4_{a+4})^+) \geq b_{2k}(P_{a+2} \cup (P^4_{n-7-a})^+)$ and there exists at least one index k such that the above inequality is strict. Then $P_{n-9-a} \cup (P^4_{a+4})^+ \succ P_{a+2} \cup (P^4_{n-7-a})^+$.

(3) If a is odd and $1 \leq a \leq \lfloor \frac{n-12}{2} \rfloor$, then by Lemma 2.4 we obtain

$$m(P_a \cup P_{n-12-a}, k-2) \leq m(P_{a-1} \cup P_{n-11-a}, k-2),$$

and

$$m(P_a \cup P_{n-12-a}, k-3) \leq m(P_{a-1} \cup P_{n-11-a}, k-3).$$

Furthermore, there exists at least one index k such that the above inequality is strict. Therefore, we have $b_{2k}(P_{n-9-a} \cup (P^4_{a+4})^+) \leq b_{2k}(P_{a+2} \cup (P^4_{n-7-a})^+)$ and there exists at least one index k such that the above inequality is strict. Thus, $P_{n-9-a} \cup (P^4_{a+4})^+ \prec P_{a+2} \cup (P^4_{n-7-a})^+$. □

From Lemmas 3.1–3.5, we can easily obtain the following results.

Theorem 3.1 *Let $n \geq 13$ and $0 \leq k \leq \lfloor \frac{n-13}{4} \rfloor$. Then quasi-order relation in $\mathcal{D}^+_{n-5}$ are determined as follows.*

(1) $n \equiv 1 (\mathrm{mod} 4)$, $P_0 \cup (P^4_{n-5})^+ \succ P_{n-9} \cup (P^4_4)^+ \succ P_2 \cup (P^4_{n-7})^+ \succ \cdots \succ P_{2k} \cup (P^4_{n-5-2k})^+ \succ P_{n-9-2k} \cup (P^4_{4+2k})^+ \succ P_{2k+2} \cup (P^4_{n-7-2k})^+ \succ \cdots \succ P_{\frac{n-13}{2}} \cup (P^4_{\frac{n+3}{2}})^+ \succ P_{\frac{n-5}{2}} \cup (P^4_{\frac{n-5}{2}})^+ \succ P_{\frac{n-9}{2}} \cup (P^4_{\frac{n-1}{2}})^+ \succ P_{\frac{n-7}{2}} \cup (P^4_{\frac{n-3}{2}})^+ \succ$

$P_{\frac{n-11}{2}} \cup (P^4_{\frac{n+1}{2}})^+ \succ \cdots \succ P_{2k+3} \cup (P^4_{n-8-2k})^+ \succ P_{n-10-2k} \cup (P^4_{2k+5})^+ \succ P_{2k+1} \cup (P^4_{n-6-2k})^+ \succ \cdots \succ P_3 \cup (P^4_{n-8})^+ \succ P_{n-10} \cup (P^4_5)^+ \succ P_1 \cup (P^4_{n-6})^+$.

(2) $n \equiv 3(\mathrm{mod}4)$, $P_0 \cup (P^4_{n-5})^+ \succ P_{n-9} \cup (P^4_4)^+ \succ P_2 \cup (P^4_{n-7})^+ \succ \cdots \succ P_{2k} \cup (P^4_{n-5-2k})^+ \succ P_{n-9-2k} \cup (P^4_{4+2k})^+ \succ P_{2k+2} \cup (P^4_{n-7-2k})^+ \succ \cdots \succ P_{\frac{n-11}{2}} \cup (P^4_{\frac{n+1}{2}})^+ \succ P_{\frac{n-7}{2}} \cup (P^4_{\frac{n-3}{2}})^+ \succ P_{\frac{n-9}{2}} \cup (P^4_{\frac{n-1}{2}})^+ \succ P_{\frac{n-5}{2}} \cup (P^4_{\frac{n-5}{2}})^+ \succ P_{\frac{n-13}{2}} \cup (P^4_{\frac{n+3}{2}})^+ \succ \cdots \succ P_{2k+3} \cup (P^4_{n-8-2k})^+ \succ P_{n-10-2k} \cup (P^4_{2k+5})^+ \succ P_{2k+1} \cup (P^4_{n-6-2k})^+ \succ \cdots \succ P_3 \cup (P^4_{n-8})^+ \succ P_{n-10} \cup (P^4_5)^+ \succ P_1 \cup (P^4_{n-6})^+$.

(3) $n \equiv 2(\mathrm{mod}4)$, $P_0 \cup (P^4_{n-5})^+ \succ P_{n-9} \cup (P^4_4)^+ \succ P_2 \cup (P^4_{n-7})^+ \succ \cdots \succ P_{2k} \cup (P^4_{n-5-2k})^+ \succ P_{n-9-2k} \cup (P^4_{4+2k})^+ \succ P_{2k+2} \cup (P^4_{n-7-2k})^+ \succ \cdots \succ P_{\frac{n-10}{2}} \cup (P^4_{\frac{n}{2}})^+ \succ P_{\frac{n-8}{2}} \cup (P^4_{\frac{n-2}{2}})^+ \succ P_{\frac{n-6}{2}} \cup (P^4_{\frac{n-4}{2}})^+ \succ P_{\frac{n-12}{2}} \cup (P^4_{\frac{n+2}{2}})^+ \succ \cdots \succ P_{2k+3} \cup (P^4_{n-8-2k})^+ \succ P_{n-10-2k} \cup (P^4_{2k+5})^+ \succ P_{2k+1} \cup (P^4_{n-6-2k})^+ \succ \cdots \succ P_3 \cup (P^4_{n-8})^+ \succ P_{n-10} \cup (P^4_5)^+ \succ P_1 \cup (P^4_{n-6})^+$.

(4) $n \equiv 0(\mathrm{mod}4)$, $P_0 \cup (P^4_{n-5})^+ \succ P_{n-9} \cup (P^4_4)^+ \succ P_2 \cup (P^4_{n-7})^+ \succ \cdots \succ P_{2k} \cup (P^4_{n-5-2k})^+ \succ P_{n-9-2k} \cup (P^4_{4+2k})^+ \succ P_{2k+2} \cup (P^4_{n-7-2k})^+ \succ \cdots \succ P_{\frac{n-12}{2}} \cup (P^4_{\frac{n+2}{2}})^+ \succ P_{\frac{n-6}{2}} \cup (P^4_{\frac{n-4}{2}})^+ \succ P_{\frac{n-8}{2}} \cup (P^4_{\frac{n-2}{2}})^+ \succ P_{\frac{n-10}{2}} \cup (P^4_{\frac{n}{2}})^+ \succ \cdots \succ P_{2k+3} \cup (P^4_{n-8-2k})^+ \succ P_{n-10-2k} \cup (P^4_{2k+5})^+ \succ P_{2k+1} \cup (P^4_{n-6-2k})^+ \succ \cdots \succ P_3 \cup (P^4_{n-8})^+ \succ P_{n-10} \cup (P^4_5)^+ \succ P_1 \cup (P^4_{n-6})^+$.

Theorem 3.2 *Let $n \geq 13$ and $0 \leq k \leq \lfloor \frac{n-13}{4} \rfloor$, then we have the following quasi-order relation in $\mathcal{C}_n^+$:*

(1) $n \equiv 1(\mathrm{mod}4)$, $B^{+,+}_{4,4}(0, n-9) \succ B^{+,+}_{4,4}(n-9, 0) \succ B^{+,+}_{4,4}(2, n-11) \succ \cdots B^{+,+}_{4,4}(2k, n-9-2k) \succ B^{+,+}_{4,4}(n-9-2k, 2k) \succ B^{+,+}_{4,4}(2k+2, n-11-2k) \succ \cdots \succ B^{+,+}_{4,4}(\frac{n-13}{2}, \frac{n-5}{2}) \succ B^{+,+}_{4,4}(\frac{n-5}{2}, \frac{n-13}{2}) \succ B^{+,+}_{4,4}(\frac{n-9}{2}, \frac{n-9}{2}) \succ B^{+,+}_{4,4}(\frac{n-7}{2}, \frac{n-11}{2}) \succ B^{+,+}_{4,4}(\frac{n-11}{2}, \frac{n-7}{2}) \succ \cdots \succ B^{+,+}_{4,4}(2k+3, n-12-2k) \succ B^{+,+}_{4,4}(n-10-2k, 2k+1) \succ B^{+,+}_{4,4}(2k+1, n-10-2k) \succ \cdots \succ B^{+,+}_{4,4}(3, n-12) \succ B^{+,+}_{4,4}(n-10, 1) \succ B^{+,+}_{4,4}(1, n-10)$.

(2) $n \equiv 3(\mathrm{mod}4)$, $B^{+,+}_{4,4}(0, n-9) \succ B^{+,+}_{4,4}(n-9, 0) \succ B^{+,+}_{4,4}(2, n-11) \succ \cdots B^{+,+}_{4,4}(2k, n-9-2k) \succ B^{+,+}_{4,4}(n-9-2k, 2k) \succ B^{+,+}_{4,4}(2k+2, n-11-2k) \succ \cdots \succ B^{+,+}_{4,4}(\frac{n-11}{2}, \frac{n-7}{2}) \succ B^{+,+}_{4,4}(\frac{n-7}{2}, \frac{n-11}{2}) \succ B^{+,+}_{4,4}(\frac{n-9}{2}, \frac{n-9}{2}) \succ B^{+,+}_{4,4}(\frac{n-5}{2}, \frac{n-13}{2}) \succ B^{+,+}_{4,4}(\frac{n-13}{2}, \frac{n-5}{2}) \succ \cdots \succ B^{+,+}_{4,4}(2k+3, n-12-2k) \succ B^{+,+}_{4,4}(n-10-2k, 2k+1) \succ B^{+,+}_{4,4}(2k+1, n-10-2k) \succ \cdots \succ B^{+,+}_{4,4}(3, n-12) \succ B^{+,+}_{4,4}(n-10, 1) \succ B^{+,+}_{4,4}(1, n-10)$.

(3) $n \equiv 2(\mathrm{mod}4)$, $B^{+,+}_{4,4}(0, n-9) \succ B^{+,+}_{4,4}(n-9, 0) \succ B^{+,+}_{4,4}(2, n-11) \succ \cdots B^{+,+}_{4,4}(2k, n-9-2k) \succ B^{+,+}_{4,4}(n-9-2k, 2k) \succ B^{+,+}_{4,4}(2k+2, n-11-2k) \succ \cdots \succ B^{+,+}_{4,4}(\frac{n-10}{2}, \frac{n-8}{2}) \succ B^{+,+}_{4,4}(\frac{n-8}{2}, \frac{n-10}{2}) \succ B^{+,+}_{4,4}(\frac{n-6}{2}, \frac{n-12}{2}) \succ B^{+,+}_{4,4}(\frac{n-12}{2}, \frac{n-6}{2}) \succ \cdots \succ B^{+,+}_{4,4}(2k+3, n-12-2k) \succ B^{+,+}_{4,4}(n-10-2k, 2k+1) \succ B^{+,+}_{4,4}(2k+1, n-10-2k) \succ \cdots \succ B^{+,+}_{4,4}(3, n-12) \succ B^{+,+}_{4,4}(n-10, 1) \succ B^{+,+}_{4,4}(1, n-10)$.

(4) $n \equiv 0(\mathrm{mod} 4)$, $B_{4,4}^{+,+}(0,n-9) \succ B_{4,4}^{+,+}(n-9,0) \succ B_{4,4}^{+,+}(2,n-11) \succ \cdots B_{4,4}^{+,+}(2k,n-9-2k) \succ B_{4,4}^{+,+}(n-9-2k,2k) \succ B_{4,4}^{+,+}(2k+2,n-11-2k) \succ \cdots \succ B_{4,4}^{+,+}(\frac{n-12}{2},\frac{n-6}{2}) \succ B_{4,4}^{+,+}(\frac{n-6}{2},\frac{n-12}{2}) \succ B_{4,4}^{+,+}(\frac{n-8}{2},\frac{n-10}{2}) \succ B_{4,4}^{+,+}(\frac{n-10}{2},\frac{n-8}{2}) \succ \cdots \succ B_{4,4}^{+,+}(2k+3,n-12-2k) \succ B_{4,4}^{+,+}(n-10-2k,2k+1) \succ B_{4,4}^{+,+}(2k+1,n-10-2k) \succ \cdots \succ B_{4,4}^{+,+}(3,n-12) \succ B_{4,4}^{+,+}(n-10,1) \succ B_{4,4}^{+,+}(1,n-10)$.

Proof. Let $e=uv$ be the edge on the cycle C_4^+ of $B_{4,4}^{+,+}(a,n-9-a)$, and $e_1=u_1v_1$ be the edge on the path of $G_4^+(a,n-9-a)$ which connects P_4 and $(P_{n-4}^4)^+$. By Lemma 2.2 we have that

$$\begin{aligned}
&b_{2k}(B_{4,4}^{+,+}(a,n-9-a))\\
=&b_{2k}(G_4^+(a,n-9-a))+b_{2k-2}(P_2\cup(P_{n-4}^4)^+)+2b_{2k-4}((P_{n-4}^4)^+)\\
=&b_{2k}((P_{n-4}^4)^+\cup P_4)+b_{2k-2}(P_3\cup P_a\cup(P_{n-5-a}^4)^+)\\
&+b_{2k-2}(P_2\cup(P_{n-4}^4)^+)+2b_{2k-4}((P_{n-4}^4)^+),
\end{aligned}$$

Obviously, we just need to consider the quasi-order in $\mathcal{D}_{n-5}^+$. Then by Theorem 3.1 we can get the results. □

4 Oriented bicyclic graph with the first five largest skew energies

In this section, we determine the oriented bicyclic graphs with the first five largest skew energies. With the help of the ordering of skew energy of $\mathcal{C}_n^+$ in Section 3, we focus on the graph $B_{4,4}^{+,+}(4,n-13)$. We need the following lemmas.

Lemma 4.1 *[19] For any bicyclic graph G with $t \le 1$, $G^{*,*} \preceq G^{+,+}$.*

Lemma 4.2 *[19]* $P_a \cup (P_{n-a}^b)^+ \prec P_2 \cup (P_{n-2}^4)^+, a \neq 2$.

Lemma 4.3 *[19]* $m(P_{n-2},k-1) \ge m(P_{n-4},k-2) \ge \cdots \ge m(P_{n-2\ell},k-\ell)$.

Theorem 4.1 *[19] Among all oriented bicyclic graphs with order $n \ge 8$, $B_{4,4}^{+,+}(0,n-9)$ has the maximal skew energy.*

We are now in the stage to get the main results in this paper.

Lemma 4.4 *Let G^σ be an oriented bicycle graph of order n with $t \le 1$, $G^\sigma \notin \mathcal{C}_n^+$. Then $G^\sigma \prec B_{4,4}^{+,+}(4,n-13)$ for $n \ge 13$.*

Proof. (i) We first consider $t = 1$. We can choose the edge $e = uv$ on C_x such that u is the common vertex of two cycles, and $G^\sigma - e \neq (P_n^4)^+, U_4^+(2, n-7)$. Obviously, $G^\sigma - e$ is a unicyclic graph and $G^\sigma - u - v$ is a forest. By Lemmas 2.1 and 2.3 we get

$$\begin{aligned}
& b_{2k}(B_{4,4}^{+,+}(4, n-13)) \\
=& m(B_{4,4}(4, n-13), k) + 2m(U_4(4, n-13), k-2) + 2m(P_{n-4}^4, k-2) \\
& + 4m(P_{n-8}, k-4) \\
=& m(U_4(4, n-9), k) + m(P_2 \cup U_4(4, n-13), k-1) \\
& + 2m(U_4(4, n-13), k-2) + 2m(P_{n-4}^4, k-2) + 4m(P_{n-8}, k-4) \\
=& m(U_4(4, n-9), k) + m(U_4(4, n-13), k-1) \\
& + 3m(U_4(4, n-13), k-2) + 2m(P_{n-4}^4, k-2) + 4m(P_{n-8}, k-4).
\end{aligned}$$

By Lemmas 2.2 and 4.1 we have that

$$\begin{aligned}
& b_{2k}(G^\sigma) \leq b_{2k}(G^{+,+}) \\
\leq & b_{2k}(G^{+,+} - e) + b_{2k-2}(G^{+,+} - u - v) + 2b_{2k-x}(G^{+,+} - C_x^+) \\
< & b_{2k}(U_4^+(4, n-9)) + b_{2k-2}(P_2 \cup P_{n-4}) + 2m(P_{n-4}, k-2) \\
= & m(U_4(4, n-9), k) + m(P_2 \cup P_{n-4}, k-1) + 4m(P_{n-4}, k-2) \\
\leq & m(U_4(4, n-9), k) + m(U_4(4, n-13), k-1) \\
& + 3m(U_4(4, n-13), k-2) + 2m(P_{n-4}^4, k-2) + 4m(P_{n-8}, k-4) \\
= & b_{2k}(B_{4,4}^{+,+}(4, n-13)).
\end{aligned}$$

(ii) If $t = 0$, then we can choose the edge $e = uv$ on C_x such that u is a vertex in a path which connects C_x and C_y with $G^\sigma - e \neq (P_n^4)^+, U_4^+(2, n-7)$. Obviously, $G^\sigma - e$ is a unicyclic graph and $G^\sigma - u - v$ is the disjoint union of a forest and a unicyclic graph. The following two cases are distinguished.

Case 1: If $x = y = 4$ and $G^\sigma \notin \{B_{4,4}^{+,+}(0, n-9), B_{4,4}^{+,+}(n-9, 0), B_{4,4}^{+,+}(2, n-11), B_{4,4}^{+,+}(n-11, 2), B_{4,4}^{+,+}(4, n-13)\}$, then by Lemmas 2.2 and 4.1 we have that

$$\begin{aligned}
& b_{2k}(G^\sigma) \leq b_{2k}(G^{+,+}) \\
\leq & b_{2k}(G^{+,+} - e) + b_{2k-2}(G^{+,+} - u - v) + 2b_{2k-4}(G^{+,+} - C_4^+) \\
< & b_{2k}(U_4^+(4, n-9)) + b_{2k-2}(P_2 \cup U_4^+(4, n-13)) \\
& + 2m(U_4(4, n-13), k-2) + 4m(P_n - 8, k-4) \\
= & m(U_4(4, n-9), k) + 2m(P_{n-4}, k-2) + m(P_2 \cup U_4(4, n-13), k-1) \\
& + 2m(P_2 \cup P_{n-8}, k-3) + 2m(U_4(4, n-13), k-2) \\
& + 4m(P_n - 8, k-4)
\end{aligned}$$

$$
\begin{aligned}
=&m(U_4(4,n-9),k)+m(U_4(4,n-13),k-1)\\
&+3m(U_4(4,n-13),k-2)+2m(P_{n-4}^4,k-2)+4m(P_{n-8},k-4)\\
=&b_{2k}(B_{4,4}^{+,+}(4,n-13)).
\end{aligned}
$$

Case 2: There is at most one cycle of length 4. We can choose the edge $e=uv$ such that $G-e$ contains no C_4. Then by Lemmas 2.2 and 4.3 we have

$$
\begin{aligned}
&b_{2k}(G^\sigma)\leq b_{2k}(G^{+,+})\\
\leq&b_{2k}(G^{+,+}-e)+b_{2k-2}(G^{+,+}-u-v)+2b_{2k-x}(G^{+,+}-C_x^+)\\
\leq&b_{2k}(G^{+,+}-e)+b_{2k-2}(G^{+,+}-u-v)+2b_{2k-x}(U_4^+(7,n-x-12))\\
<&b_{2k}(U_4^+(4,n-9))+b_{2k-2}(P_2\cup U_4^+(7,n-16))\\
&+2m(U_4(7,n-16),k-2)+4m(P_n-8,k-4)\\
=&m(U_4(4,n-9),k)+2m(P_{n-4},k-2)+m(P_2\cup U_4(7,n-16),k-1)\\
&+2m(P_2\cup P_{n-8},k-3)+2m(U_4(4,n-13),k-2)\\
&+4m(P_n-8,k-4)\\
\leq&m(U_4(4,n-9),k)+m(U_4(4,n-13),k-1)\\
&+3m(U_4(4,n-13),k-2)+2m(P_{n-4}^4,k-2)+4m(P_{n-8},k-4)\\
=&b_{2k}(B_{4,4}^{+,+}(4,n-13)).
\end{aligned}
$$

Combining the above two cases, we complete the proof. □

Lemma 4.5 *Let G^σ be an oriented bicycle graph of order n with $t\geq 2$ and $G^\sigma\notin\mathcal{C}_n^+$. Then $G^\sigma\prec B_{4,4}^{+,+}(4,n-13)$ for $n\geq 13$.*

Proof. We prove the statement by dividing into four cases.

Case 1: $x=y=z=4$. Then $t=3$. If both C_x and C_y are oddly oriented, then C_z must be evenly oriented. We can choose the edge $e=uv$ such that u is the common vertex of C_x, C_y and $G^\sigma-e\neq(P_n^4)^+, U_4^+(2,n-7)$. Without loss of generality, set $e\in C_y$. So, $G^\sigma-C_x=G^\sigma-e-C_x$. Then

$$
\begin{aligned}
&b_{2k}(G^{+,+,-})\\
=&m(G,k)+2m(G-C_x,k-2)+2m(G-C_y,k-2)\\
&-2m(G-C_z,k-2)\\
\leq&m(G-e,k)+m(G-u-v,k-1)+2m(G-e-C_x,k-2)\\
&+2m(G-C_y,k-2)\\
<&b_{2k}(G^\sigma-e)+m(P_{n-2},k-1)+2m(P_{n-4},k-2)
\end{aligned}
$$

$$
\begin{aligned}
&\leq b_{2k}(U_4^+(4,n-9)) + m(P_{n-2},k-1) + 2m(P_{n-4},k-2)\\
&= m(U_4(4,n-9),k) + m(P_{n-2},k-1) + 4m(P_{n-4},k-2)\\
&\leq m(U_4(4,n-9),k) + m(U_4(4,n-13),k-1)\\
&\quad + 3m(U_4(4,n-13),k-2) + 2m(P_{n-4}^4,k-2) + 4m(P_{n-8},k-4)\\
&= b_{2k}(B_{4,4}^{+;+}(4,n-13)).
\end{aligned}
$$

If either C_x or C_y is oddly oriented, then C_z must be oddly oriented. If both C_x and C_y are evenly oriented, then C_z is also evenly oriented. Similarly, we can prove that $b_{2k}(G^\sigma) < b_{2k}(B_{4,4}^{+;+}(4,n-13))$.

Case 2: $x = y = 4,\ z \neq 4$. Then $t = 2$ and $z = 6$. If both C_x and C_y are oddly oriented, then C_z is oddly oriented. Since $n \geq 13$, we can choose the edge $e = uv$ such that u is the common vertices of C_x and C_y but v is not and $G-u-v$ is acyclic. Clearly, we can make $G^\sigma - e \notin \{(P_n^4)^+, U_4^+(2,n-7)\}$. Without loss of generality, let $e \in C_y$. Note that $G^\sigma - C_x = G^\sigma - e - C_x$ and $G^\sigma - C_y$, $G^\sigma - C_z$ is acyclic. Then

$$
\begin{aligned}
&b_{2k}(G^{+,+,+})\\
&= m(G,k) + 2m(G-C_x,k-2) + 2m(G-C_y,k-2)\\
&\quad + 2m(G-C_z,k-3)\\
&\leq m(G-e,k) + 2m(G-e-C_x,k-2) + m(G-u-v,k-1)\\
&\quad + 2m(P_{n-4},k-2) + 2m(P_{n-6},k-3)\\
&< b_{2k}(G^\sigma - e) + m(P_{n-2},k-1) + 2m(P_{n-4},k-2)\\
&\quad + 2m(P_{n-6},k-3)\\
&\leq b_{2k}(U_4^+(4,n-9)) + m(P_{n-2},k-1) + 2m(P_{n-4},k-2)\\
&\quad + 2m(P_{n-6},k-3)\\
&= m(U_4(4,n-9),k) + m(P_{n-2},k-1) + 4m(P_{n-4},k-2)\\
&\quad + 2m(P_{n-6},k-3)\\
&\leq m(U_4(4,n-9),k) + m(U_4(4,n-13),k-1)\\
&\quad + 3m(U_4(4,n-13),k-2) + 2m(P_{n-4}^4,k-2) + 4m(P_{n-8},k-4)\\
&= b_{2k}(B_{4,4}^{+;+}(4,n-13)).
\end{aligned}
$$

If either C_x or C_y is oddly oriented, then C_z is evenly oriented. If both C_x and C_y are evenly oriented, then C_z is oddly oriented. We can also prove that $b_{2k}(G^\sigma) < b_{2k}(B_{4,4}^{+;+}(4,n-13))$.

Case 3: If $x = 4,\ z \geq y \geq 5$, we can choose the edge $e = uv$ on C_x satisfying that u is the common vertex of C_x and C_y. Obviously, $G^\sigma - u$ is acyclic and $G^\sigma - e \notin \{(P_n^4)^+, U_4^+(2,n-7),\ G^\sigma - C_y = G^\sigma - e - C_y\}$. Then

$$
b_{2k}(G^\sigma)
$$

$$
\begin{aligned}
&\le m(G,k)+2m(G-C_x,k-\frac{x}{2})+2m(G-C_y,k-\frac{y}{2})\\
&\quad+2m(G-C_z,k-\frac{z}{2})\\
&\le m(G-e,k)+2m(G-e-C_y,k-\frac{y}{2})+m(G-u-v,k-1)\\
&\quad+2m(P_{n-4},k-2)+2m(P_{n-6},k-3)\\
&\le b_{2k}(G^\sigma-e)+m(P_{n-2},k-1)+2m(P_{n-4},k-2)\\
&\quad+2m(P_{n-6},k-3)\\
&< b_{2k}(U_4^+(7,n-12))+m(P_{n-2},k-1)+2m(P_{n-4},k-2)\\
&\quad+2m(P_{n-6},k-3)\\
&= m(U_4(7,n-12),k)+m(P_{n-2},k-1)+4m(P_{n-4},k-2)\\
&\quad+2m(P_{n-6},k-3)\\
&\le m(U_4(4,n-9),k)+m(U_4(4,n-13),k-1)\\
&\quad+3m(U_4(4,n-13),k-2)+2m(P^4_{n-4},k-2)+4m(P_{n-8},k-4)\\
&= b_{2k}(B^{+;+}_{4,4}(4,n-13)).
\end{aligned}
$$

Case 4: If there is no cycle of length 4, then $z\ge y\ge 5$, $z\ge x\ge 5$. We can choose the edge $e=uv$ on C_y such that u is the common vertex of C_x and C_y. Note that $G^\sigma-e\notin\{(P_n^4)^+,\ U_4^+(2,n-7)\}$ and $G^\sigma-C_x=G^\sigma-e-C_x$. Then

$$
\begin{aligned}
&b_{2k}(G^\sigma)\\
&\le m(G,k)+2m(G-C_x,k-\frac{x}{2})+2m(G-C_y,k-\frac{y}{2})\\
&\quad+2m(G-C_z,k-\frac{z}{2})\\
&\le m(G-e,k)+2m(G-e-C_x,k-3)\\
&\quad+m(G-u-v,k-1)+4m(P_{n-6},k-3)\\
&\le b_{2k}(G^\sigma-e)+m(P_{n-2},k-1)+4m(P_{n-6},k-3)\\
&< b_{2k}(U_4^+(7,n-12))+m(P_{n-2},k-1)+4m(P_{n-6},k-3)\\
&= m(U_4(7,n-12),k)+m(P_{n-2},k-1)+2m(P_{n-4},k-2)\\
&\quad+4m(P_{n-6},k-3)\\
&\le m(U_4(4,n-9),k)+m(U_4(4,n-13),k-1)\\
&\quad+3m(U_4(4,n-13),k-2)+2m(P^4_{n-4},k-2)+4m(P_{n-8},k-4)\\
&= b_{2k}(B^{+;+}_{4,4}(4,n-13)).
\end{aligned}
$$

Combining all these cases above, we complete the proof. □

By Lemma 4.4, 4.5 and Theorem 3.2, we obtain the following main result.

Theorem 4.2 *Among all oriented bicyclic graphs with order* $n \geq 13$*, the graphs* $B_{4,4}^{+,+}(0, n-9) \succeq B_{4,4}^{+,+}(n-9, 0) \succeq B_{4,4}^{+,+}(2, n-11) \succeq B_{4,4}^{+,+}(n-11, 2) \succeq B_{4,4}^{+,+}(4, n-13)$ *have the first five largest skew energies.*

References

[1] C. Adiga, R. Balakrishnan and W. So. The skew energy of a digraph. *Linear Algebra Appl.,* **432** (2010) 1825–1835.

[2] Ş. B. Bozkurt and D. Bozkurt. On incidence energy. *MATCH Commun. Math. Comput. Chem.,* **72** (2014) 215–225.

[3] Ş. B. Bozkurt, I. Gutman. Estimating the incidence energy. *MATCH Commun. Math. Comput. Chem.,* **73** (2013) 143–156.

[4] D. Cvetković, M. Doob and H. Sachs. Spectra of graphs. Academic Press, New York, 1979.

[5] K.C. Das, S. Sorgun. On Randić energy of graphs. *MATCH Commun. Math. Comput. Chem.,* **72** (2014) 227–238.

[6] S. Gong, G. Xu. 3-regular digraphs with optimum skew energy. *Linear Multilinear Algebra,* **436** (2012) 465–471.

[7] I. Gutman. The energy of a graph. *Ber. Math.-Statist. Sekt. Forsch. Graz,* **103** (1978) 1–22.

[8] I. Gutman. The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.). *Algebraic Combinatorics and Applications,* Springer, Berlin, 2001, pp. 196–211.

[9] I. Gutman, X. Li, J. Zhang. Graph energy, in: M. Dehmer, F.Emmert-Streb(Eds.). *Analysis of Complex Networks: From Biology to Linguistics,* Wiley- VCH Verlag, Weinheim. 2009, pp. 145–174.

[10] I. Gutman, O.E. Polansky. Mathematical Concepts in Organic Chemistry. Springer-Verlag, Berlin, 1986.

[11] J. Shao, F. Gong, I. Gutman. New apporaches for the real and complex integral formulas of the energy of a polynomial. *MATCH Commun. Math. Comput. Chem.,* **66** (2011) 849–861.

[12] Y. Hou and T. Lei. Characteristic polynomials of skew-adjacency matices of oriented graphs. *Electron. J. Combin.,* **18**(1) (2011) R156.

[13] Y. Hou, X. Shen and C. Zhang. Oriented unicyclic graphs with extremal skew energy. http://arXiv.org/abs/1108.6229.

[14] X. Li, Y. Shi, I. Gutman. *Graph energy.* Springer, New York, 2012.

[15] X. Li, J. Wang. Randić energy and Randić eigenvalues. *MATCH Commun. Math. Comput. Chem.,* accepted.

[16] X. Li, Y. Shi, M. Wei and J. Li. On a conjecture about tricyclic graphs with maximal energy. *MATCH Commun. Math. Comput. Chem.,* **72**(1) (2014) 183–214.

[17] M. Mateljevic, V. Bozin, I. Gutman. Energy of a polynomial and the Coulson integral formula. *J. Math. Chem.,* **48** (2010) 1062–1068.

[18] I.Z. Milovanovic, E.I. Milovanovic, A. Zakic. A short note on graph energy. *MATCH Commun. Math. Comput. Chem.,* **72** (2014) 179–182.

[19] X. Shen, Y. Hou and C. Zhang. Bicyclic digraphs with extremal skew energy. *Electron. J. Linear Algebra,* **23** (2012) 340–355.

[20] B. Shader and W.So. Skew spectra of oriented graphs. *Electron. J. Combin.,* **16** (2009) 1–6.

[21] H. Shan and J.Shao. Graph energy change due to edge grafting operations and its application. *MATCH Commun. Math. Comput. Chem.,* **64** (2010) 25–40.

[22] D. Stevanovic, M. Milosevic, P. Hic, M. Pokorny. Proof of a conjecture on distance energy of complete multipartite graphs. *MATCH Commun. Math. Comput. Chem.,* **70** (2013) 157–162.

[23] G. Tian. On the skew energy of orientations of hypercubes. *Linear Algebra Appl.,* **435** (2011) 2140–2149.

[24] J.F. Wang, L. Zhao, C.F. Ye. Bicyclic oriented graphs with the second largest skew-energy. *Linear Algebra Appl.,* **459** (2014) 43–57.

[25] J. Zhu. Oriented unicyclic graphs with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies. *Linear Algebra Appl.,* **437** (2012) 2630–2649.