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Vertex partitions of r-edge-colored graphs

JIN Ze-min[1,2] LI Xue-liang[1]

Abstract. Let G be an edge-colored graph. The monochromatic tree partition problem is to find the minimum number of vertex disjoint monochromatic trees to cover the all vertices of G. In the authors' previous work, it has been proved that the problem is NP-complete and there does not exist any constant factor approximation algorithm for it unless $P = NP$. In this paper the authors show that for any fixed integer $r \geq 5$, if the edges of a graph G are colored by r colors, called an r-edge-colored graph, the problem remains NP-complete. Similar result holds for the monochromatic path (cycle) partition problem. Therefore, to find some classes of interesting graphs for which the problem can be solved in polynomial time seems interesting. A linear time algorithm for the monochromatic path partition problem for edge-colored trees is given.

§1 Introduction

Many graph partition problems and their corresponding computational complexity problems have been well studied in [3,5], most of which were shown to be NP-complete. A list of graph partition problems can be found in the book [5].

Some researchers also focused on graph partition problems in edge-colored graphs[4,6−8,11]. The aim is to find some kind of vertex disjoint monochromatic subgraphs (e.g. trees, cycles, or paths) to cover all the vertices of the given graph. Motivated by the results of [4,7,8,11], Jin and Li[9] considered the following problems: Given an edge-colored graph G, find the minimum number of vertex disjoint monochromatic trees, cycles, paths, respectively, which cover the vertices of G. For convenience, these three problems are addressed as PGMT, PGMC, PGMP problem, respectively. Jin and Li[9] showed that all these three problems are NP-hard and there does not exist constant factor approximation algorithm for any of these three problems unless $P = NP$.

Note that PGMT problem looks like the problem of partitioning a graph into induced forests[5]. But actually it is not the case. The following facts are easily seen. If G is colored

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properly, i.e., any adjacent edges receive different colors, both PGMT and PGMP problems are equivalent to the maximum matching problem which can be solved in polynomial time[5]. If G is colored with one color, PGMT problem is equivalent to the spanning tree problem and can be solved in polynomial time, while PGMC and PGMP problems are equivalent to the Hamiltonian cycle and Hamiltonian path problems, respectively, and both of them are NP-complete. Jin and Li[9] asked the following problem: Does PGMT (PGMC, or PGMP) problem remain to be NP-complete when G is colored with two colors? Jin and Li[10] solved the problems for 2-edge-colored complete multipartite graphs. In this paper we show that for any fixed integer $r \geq 5$, if the edges of a graph G are colored by r colors(called an r-edge-colored graph), these three problems remain to be NP-complete.

Note that the problem of partitioning edge-colored graphs by vertex disjoint monochromatic subgraphs (e.g. paths) seems to be more difficult than the corresponding problem in uncolored graphs. Though the path partition problem (i.e., to partition graphs into minimum number of vertex disjoint paths which cover all the vertices of the given graphs) can be solved in polynomial time for some classes of special graphs[1,2,12−14], it is open for the monochromatic path partition problem. So, it becomes an interesting question to identify graph classes for which there exist polynomial time algorithms for solving the above problems. In §3, we give a linear time algorithm to the monochromatic path partition problem for edge-colored trees.

§2 NP-completeness results

At first, we focus on PGMT problem. The corresponding decision version is defined formally as follows.

PGMT Problem

Instance: An edge-colored graph G and a positive integer k.

Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G ?

The decision versions of PGMC and PGMP problems can be defined in a similar way. Let r be a fixed positive integer. If G is an r-edge-colored graph, we call the corresponding PGMT problem as r-PGMT problem. The r-PGMC and r-PGMP problems can be defined similarly. Here we have the following NP-completeness results stronger than those in [9], which are based on the results that the vertex cover problem is NP-complete for any planar graphs and the famous 4-color Theorem.

Theorem 2.1. For any fixed integer $r \geq 5$, the r-PGMT problem remains NP-complete.

Proof. The problem is clear in NP, since a nondeterministic algorithm needs only to guess a set of trees and check in polynomial time that the trees in the set are vertex disjoint monochromatic ones and cover the vertices of the given graph.

Now we transform the vertex cover problem for planar graphs (which is NP-complete[5]) to PGMT problem. Let an arbitrary instance of the vertex cover problem for planar graphs be given by a graph H. Here we construct an r-edge-colored graph G such that there are k or less vertices of H which cover all the edges of H if and only if G contains $k+1$ or less vertex disjoint monochromatic trees which cover the vertices of G.

Let $V(H) = \{v_1, v_2, \cdots, v_n\}$ and $E(H) = \{e_1, e_2, \cdots, e_m\}$. The graph G is constructed as follows:

$$V(G) = \{v_1, v_2, \cdots, v_n, e_1, e_2, \cdots, e_m\},\ E(G) = E_1 \cup E_2$$

where $E_1 = \{v_iv_j, 1 \leq i \neq j \leq n\},\ E_2 = \{v_ie_j : v_i \text{ is incident to } e_j \text{ in } H\}$.

Without loss of generality, we may assume that $|V(H)|$ is large enough. Then, since $r \geq 5$ and H is planar, by the famous 4-Color Theorem we have $V(H) = V_1 \cup V_2 \cup \cdots \cup V_{r-1}$, where $V_i \cap V_j = \emptyset$ for any $i \neq j$ and $V_i, i = 1, 2, \cdots, r-1$, is a nonempty independent set of H.

Color the edges of G by r colors $c_1, c_2, \cdots, c_r$ as follows. At first, color all the edges of E_1 by the color c_r. Let $v_ie_j \in E(G)$ and $v_i \in V_l, 1 \leq l \leq r-1$. Then color the edge v_ie_j by the color c_l. It is easy to see that the construction can be accomplished in polynomial time and the edges of G are colored by r colors. We claim that there are k or less vertices in H which cover all the edges of H if and only if G contains $k+1$ or less vertex disjoint monochromatic trees which cover the vertices of G.

If there are k vertices of H which cover all the edges of H, it is easy to see that G contains $k+1$ or less vertex disjoint monochromatic trees which cover the vertices of G.

Suppose that G contains $k+1$ vertex disjoint monochromatic trees denoted by

$$\Gamma = \{T_1, T_2, \cdots, T_{k+1}\},$$

which cover the vertices of G. From the construction and the given edge-coloring of G, each tree in Γ containing a vertex c_j contains at most one vertex v_i for some $1 \leq i \leq n$. If $k \geq n$, the claim is true. So, we assume that $k < n$.

Let $T_i,\ i = 1, 2, \cdots, t,\ t \leq k < n$, be the trees, each of which contains an edge v_pe_q for some $1 \leq p \leq n$ and $1 \leq q \leq m$. Without loss of generality, we can assume that $v_i \in T_i$, $i = 1, 2, \cdots, t$. If for any $t+1 \leq j \leq k+1$, T_j is not composed of a single vertex e_r for some $1 \leq r \leq m$, then it is easy to see that $v_i,\ i = 1, 2, \cdots, t$ cover all the edges of H. Suppose that T_{t+i}, $i = 1, 2, \cdots, t'$, is composed of a single vertex e_r for some $1 \leq r \leq m$. Without loss of generality, we can assume that $V(T_{t+i}) = \{e_i\}, i = 1, 2, \cdots, t'$. Then $t + t' \leq k < n$, since there is some vertex $v_s \notin V(T_j)$ for any $t \leq j \leq t + t'$. For each e_i, $1 \leq i \leq t'$, find a vertex v_{p_i} incident to it. It is easy to see that the vertex sets $\{v_1, v_2, \cdots, v_t, v_{p_1}, v_{p_2}, \cdots, v_{p_{t'}}\}$ form an vertex cover of H with size at most k. This completes the proof.

By the same technique, one can show that the problem of finding the minimum number of vertex disjoint monochromatic stars to cover the vertices of an r-edge-colored graph is also NP-complete. For r-PGMC problem we have the following result.

Theorem 2.2. For any fixed integer $r \geq 5$, the r-PGMC problem remains NP-complete.

Proof. The problem is clear in NP, since a nondeterministic algorithm needs only to guess a set of cycles and check in polynomial time that the cycles in the set are vertex disjoint monochromatic ones which cover the vertices of the given graph.

Now we transform the vertex cover problem for planar graphs (which is NP-complete[5]) to PGMC problem. Let an arbitrary instance of the vertex cover problem for planar graphs be given by a planar graph H. Here we construct an r-edge-colored multi-graph G such that there are k or less vertices of H which cover all the edges of H if and only if G contains $k+1$ or less vertex disjoint monochromatic cycles which cover the vertices of G.

Let $V(H) = \{v_1, v_2, \cdots, v_n\}$ and $E(H) = \{e_1, e_2, \cdots, e_m\}$. Construct the multi-graph G as follows:

$$V(G) = \{e_1, e_2, \cdots, e_m, e_1^*, e_2^*, \cdots, e_m^*, e_1^{**}, e_2^{**}, \cdots, e_m^{**}, v_1, v_2, \cdots, v_n, u, v, w\}.$$

Without loss of generality, we may assume that $|V(H)|$ is large enough. Then, since $r \geq 5$ and H is planar, by the famous 4-Color Theorem we have $V(H) = V_1 \cup V_2 \cup \cdots \cup V_{r-1}$, where $V_i \cap V_j = \emptyset$ for any $i \neq j$ and $V_i, i = 1, 2, \cdots, r-1$, is a nonempty independent set of H.

Next, construct and color the edges of G as follows:

(1) If e_i is incident to v_j in H and $v_j \in V_l$, $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq r-1$, then each of the vertices e_i, e_i^* and e_i^{**} is connected to v_j by an edge with color c_l in G.

(2) For each vertex v with $v \in V_l, 1 \leq l \leq r-1$, let $I(v) = \{i : e_i \text{ is incident to } v \text{ in } H\}$. Then construct a complete graph with vertex set $\{e_i : i \in I(v)\} \cup \{e_i^* : i \in I(v)\} \cup \{e_i^{**} : i \in I(v)\}$, whose edges are colored by the same color c_l. So, in general what we constructed is a multi-graph.

(3) Construct a complete graph with vertex set $\{v_1, v_2, \cdots, v_n, u, v, w\}$, whose edges are colored by the same color c_r.

It is easy to show that the construction can be accomplished in polynomial time and the edges of G are colored by r colors. We claim that there are k or less vertices of H which cover all the edges of H if and only if G contains $k+1$ or less vertex disjoint monochromatic cycles which cover the vertices of G.

If there are k vertices of H which cover all the edges of H, it is easy to find that G contains $k+1$ vertex disjoint monochromatic cycles which cover the vertices of G.

Suppose that G contains $k+1$ vertex disjoint monochromatic cycles denoted by $\Gamma = \{C_1, C_2, \cdots, C_{k+1}\}$, which cover the vertices of G. From the construction and the given edge-coloring of G, each cycle in Γ containing a vertex e_j contains at most one vertex v_i for some $1 \leq i \leq n$. If $k \geq n$, we are done. So, we assume that $k < n$.

Let C_i, $i = 1, 2, \cdots, t$, be the cycles containing some edges $e_p v_q$, $e_p^* v_q$, or $e_p^{**} v_q$. From the construction we have that none of u, v and w can lie in any C_i, $i = 1, 2, \cdots, t$ and each C_i, $i = 1, 2, \cdots, t$, contains a unique vertex v_q for some $1 \leq q \leq n$. Without loss of generality, we can assume that $v_i \in C_i$, $i = 1, 2, \cdots, t$. If C_j contains no vertex of $\{e_1, e_2, \cdots, e_m\}$ for any $t+1 \leq j \leq k+1$, then it is easy to see that v_i, $i = 1, 2, \cdots, t$ cover all the edges H. So, we assume that C_{t+i}, $i = 1, 2, \cdots, t'$, are the cycles, each of which only contains vertices of $\{e_1, e_2, \cdots, e_m, e_1^*, e_2^*, \cdots, e_m^*, e_1^{**}, e_2^{**}, \cdots, e_m^{**}\}$. Since none of u, v and w can lie in any C_{t+i}, $i = 1, 2, \cdots, t'$, it is easy to see that $t + t' \leq k$. Consider the cycle C_{t+i}, $i = 1, 2, \cdots, t'$. Denote the color of the cycle C_{t+i} by c_{r_i}, $i = 1, 2, \cdots, t'$. Let $I_i = \{j \mid C_{t+i} \cap \{e_j, e_j^*, e_j^{**}\} \neq \emptyset\}$. Then each edge e_j, $j \in I_i$, is incident to v_{r_i} in H. It is easy to see that the subsets $v_1, v_2, \cdots, v_t, v_{r_i}$, $i = 1, 2, \cdots, t'$ form a vertex cover of H. This completes the proof.

By the same proof technique, we can prove the following result. For simplicity, we omit the details.

Theorem 2.3. For any fixed integer $r \geq 5$, the r-PGMP problem remains NP-complete.

Remark. From the above proofs we can see that there are at most two colors appearing at each vertex in the constructed edge-colored graph G. So, as a consequence we get that all the r-PGMT, r-PGMC and r-PGMP problems are NP-complete for edge-colored graphs in which there are at most two colors appearing at each vertex.

Note that we have determined the complexity of PGMT problem for $r \geq 5$. For the rest cases, the complexity remains unknown.

§3 Linear time algorithm for edge-colored trees

In this section we present a linear time algorithm for PGMP problem with edge-colored trees. At first, we establish some edge-deletion rules and an edge-contraction rule, which are the base for our algorithm. These rules characterize the edges that can be deleted or contracted from the given tree without affecting its minimum number of vertex disjoint monochromatic paths to cover all the vertices. Applied to an initially given edge-colored tree T, the edge-deletion rules delete some vertex disjoint monochromatic paths from the current tree, which together form a set of vertex disjoint monochromatic paths. From the edge-contraction rule we can find an optimal partition of the initial tree.

Throughout this section we assume that the edge-colored tree T is arranged on the plane by BFS. For each edge e, denote its color by $c(e)$. We treat a path containing only one vertex as a monochromatic path. A monochromatic path partition is called optimal if it has the minimum number of vertex disjoint monochromatic trees which together cover all the vertices.

Suppose that the vertex u has vertices $u_1, u_2, \cdots, u_s$ as its sons, $s \geq 1$, all of which are leaves. Denote by v the father of the vertex u if there exists. We have the following lemmas.

Lemma 3.1. If no two edges incident to u have the same color, then there is an optimal monochromatic path partition $\mathcal{P}$ of T such that $\{P_1, P_2, \cdots, P_s\} \subseteq \mathcal{P}$, where $V(P_1) = \{u, u_1\}$ and $V(P_i) = \{u_i\}, i = 2, \cdots, s$. Let $H = T - \{u, u_1, u_2, \cdots, u_s\}$, a tree. Then $\mathcal{P}_H = \mathcal{P} \setminus \{P_1, P_2, \cdots, P_s\}$ is an optimal monochromatic path partition of H.

Proof. Let $\mathcal{P}$ be an optimal monochromatic path partition of T and $u \in V(P), P \in \mathcal{P}$. Since no two edges incident to u have the same color, u must be an end of the path P, and P contains at least two vertices. If $uv \in E(P)$, then $Q_i \in \mathcal{P}, V(Q_i) = \{u_i\}, i = 1, 2, \cdots, s$. By deleting the vertex u from P and replacing the path Q_1 by path P_1, we obtain the desired partition. If $uu_i \in P$ for some $1 \leq i \leq s$, then $V(P) = \{u, u_i\}$ and $Q_j \in \mathcal{P}, V(Q_j) = \{u_j\}, j \neq i$. If $i \neq 1$, by replacing the paths P and Q_1 by paths P_1 and P_i, respectively, we obtain the desired partition still denoted by $\mathcal{P}$. Let $H = T - \{u, u_i, i = 1, 2, \cdots, s\}$, a tree. It is easy to see that $\mathcal{P}_H = \mathcal{P} \setminus \{P_1, P_2, \cdots, P_s\}$ is an optimal monochromatic path partition of H.

By the same proof technique, we can prove the following two lemmas. For simplicity, we omit the details.

Lemma 3.2. If $c(uu_i) = c(uu_j)$ for some $i \neq j$, then there is an optimal monochromatic path partition $\mathcal{P}$ of T such that $P_l \in \mathcal{P}, V(P_l) = \{u_l\}, i \neq l \neq j$, and $P_{ij} \in \mathcal{P}, V(P_{ij}) = \{u_i, u, u_j\}$. Let $H = T - \{u, u_1, u_2, \cdots, u_s\}$, a tree. Then $\mathcal{P}_H = \mathcal{P} \setminus \{P_{ij}, P_l, i \neq l \neq j\}$ is an optimal monochromatic path partition of H.

Lemma 3.3 If $c(uv) = c(uu_l)$ for some $1 \le l \le s$ and $c(uu_i) \ne c(uu_j)$ for any $i \ne j$, then there is an optimal monochromatic path partition $\mathcal{P}$ of T such that $P_i \in \mathcal{P}, V(P_i) = \{u_i\}, i \ne l$. Let $H = T - \{u_i : i \ne l\}$, a tree. Then $\mathcal{P}_H = \mathcal{P} \setminus \{P_i : i \ne l\}$ is an optimal monochromatic path partition of H.

An edge e of T is said to be contracted if it is deleted and its ends are identified, the resulting graph is denoted by $T \bullet e$. Let $e \in E(T)$ and $\mathcal{P}$ be a path partition of T, let $e \in E(P), P \in \mathcal{P}$, and let $\mathcal{P} \bullet e = (\mathcal{P} \setminus \{P\}) \cup \{P \bullet e\}$. We have the following lemma.

Lemma 3.4. If $s = 1$ and $c(uv) = c(uu_1)$, then in any optimal monochromatic path partition $\mathcal{P}$ of T the edge uu_1 is contained in a path of $\mathcal{P}$, and $\mathcal{P} \bullet uu_1$ is an optimal monochromatic path partition of $T \bullet uu_1$.

Proof. Let $\mathcal{P}$ be an arbitrary optimal monochromatic path partition of T, and P be the path containing u in $\mathcal{P}$. Since $s = 1$ and $c(uv) = c(uu_1)$, P contains the edge $c(uu_1)$. Otherwise we can get a monochromatic path partition with fewer paths. It is easy to see that $\mathcal{P} \bullet uu_1$ is an optimal monochromatic path partition of $T \bullet uu_1$.

Now we give our algorithm for partitioning an edge-colored tree by monochromatic paths.

Algorithm MPPT: Find an optimal monochromatic path partition of an edge-colored tree.

Input: An edge-colored tree T.

Output: An optimal monochromatic path partition of T.

Step 1. Arrange T on the plane by BFS, and set $\mathcal{P} = \emptyset$.

Step 2. Let u be a vertex with sons $u_1, u_2, \cdots, u_s$, $s \ge 1$, all of them lie on the bottom level. If u is not the root, denote by v the father of u.

2.1. If any two edges incident to u have different colors, then do

$\mathcal{P} \leftarrow \mathcal{P} \cup \{u_i, i \ne 1\} \cup \{uu_1\}$, $T \leftarrow T - \{u, u_1, u_2, \cdots, u_s\}$;

2.2. If $c(uu_i) = c(uu_j)$ for some $i \ne j$, then do

$\mathcal{P} \leftarrow \mathcal{P} \cup \{u_l, l \ne i, l \ne j\} \cup \{u_i uu_j\}$, $T \leftarrow T - \{u, u_1, u_2, \cdots, u_s\}$;

2.3. If $c(uv) = c(uu_l)$ for some $1 \le l \le s$ and $c(uu_i) \ne c(uu_j)$ for any $i \ne j$, then do

$$\mathcal{P} \leftarrow \mathcal{P} \cup \{u_i, i \ne l\},\ T \leftarrow T - \{u_i, i \ne l\};$$

2.4. If $s = 1$ and $c(uv) = c(uu_1)$, then do $T \leftarrow T \bullet uu_1$.

Step 3. If T is not a vertex, go to Step 2. Otherwise, stop !

Theorem 3.5. Algorithm MPPT finds an optimal monochromatic path partition of an edge-colored tree in linear time.

Proof. The correctness of the algorithm follows from Lemmas 3.1 through 3.4. For the time complexity we note that it depends mainly on Steps 2.1-2.4 for some vertex with all sons on the bottom level. Since for each vertex u with all the sons in the bottom level, Steps 2.1-2.4 can be done in $O(d(u))$ time, our Algorithm MPPT runs in linear time.

Note that modifying the algorithm slightly, we can solve the monochromatic tree partition problem for edge-colored trees in linear time.

§4 Concluding remarks

The problems described above are applicable in our real world. In communication networks there are many different types of communication medium, such as optic fiber, cable, microwave, telephone line and so on. A communication vertex may communicate with different vertices by different types of communication medium. In a communication network there frequently arises a situation where some information must be communicated from some vertices to all other vertices in the network. Suppose that all communication is done subject to the following restrictions:

(1) a vertex may participate in the communication by only one medium;

(2) a vertex may only send the message to an adjacent vertex.

We focus on the problem of determining the minimum number of message originators necessary to complete the communication. This problem can be formulated as PGMP problem. The algorithm presented in §3 can be used to find in linear time the minimum number of the necessary message originators, from which message is sent to all other vertices in a tree network.

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1 Center for Combinatorics and LPMC-TJKLC, Nankai University, Tianjin 300071, China. Email: lxl@nankai.edu.cn

2 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China. Email: zeminjin@zjnu.cn